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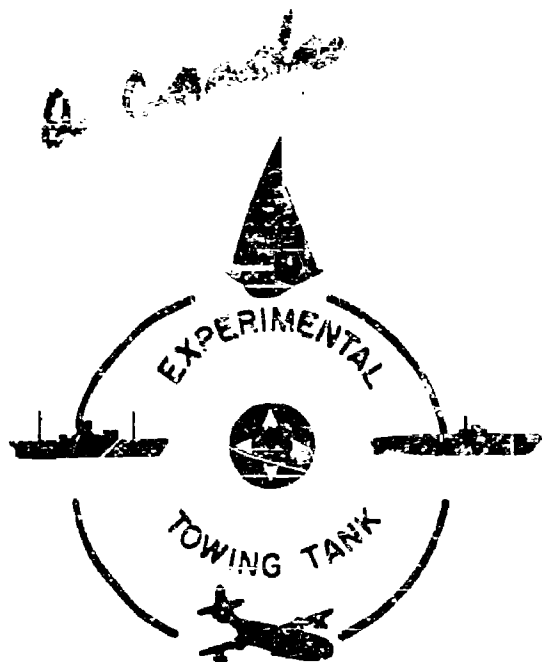
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Report No. 469
January 1959

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**DIFFERENTIAL EQUATIONS WITH
A DISCONTINUOUS FORCING TERM**

by

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**ENCLOSURE: E.T.T. Report No. 469, "Differential Equations
with a Discontinuous Forcing Term," by
Donald Wayne Bushaw.**

The enclosed report presents a thorough study of the differential equation

$$m\ddot{x} + c\dot{x} + kx = Df(x, \dot{x}) \quad ,$$

where f is a discontinuous function which assumes only the values ± 1 . The study shows how f should be defined in order that any error in (x, \dot{x}) , that is, any deviation from $(0,0)$, will vanish in a minimum of time. Curves are drawn in the $x\dot{x}$ phase plane showing where f should "switch" its value from $+1$ to -1 or vice versa. The work should be of interest to people concerned with the design of servomechanisms and similar devices.

The report was written as a dissertation for a Doctor of Philosophy, Department of Mathematics, Princeton University. The work involved, however, was a part of the research program under the Office of Naval Research, Limit Control project, Contract Nonr-26302, NR 341-009 at the Experimental Towing Tank, Stevens Institute of Technology. It is therefore being published by the latter organization as a technical report on the project.

Experimental Towing Tank
Stevens Institute of Technology
Hoboken, New Jersey

DIFFERENTIAL EQUATIONS
WITH A DISCONTINUOUS FORCING TERM

A Dissertation
Presented to the Faculty of Princeton University
In Candidacy for the Degree of Doctor of Philosophy
Recommended for Acceptance
by the Department of Mathematics
June 1952

by

Donald Wayne Bushaw

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SUMMARY

In many problems arising in connection with designing servomechanisms and similar systems one has to solve

$$m \frac{d^2x}{dt^2} + c \frac{dx}{dt} + kx = \dot{D}$$

with the discontinuous forcing term \dot{D} , and to find a transition or switching curve in the phase plane so that solutions reach the zero state in minimum time.

This problem arises in designing the systems so as to obtain optimum performance, and is therefore of considerable practical as well as mathematical interest; but very little work has previously been done on it, and that mainly from a physical rather than mathematical point of view.

In this thesis there is given a complete solution of this problem. The treatment is much more lengthy than one would expect, because it has been necessary to break the problem up into cases; none of the general methods available in the literature apply. The final result is given in Theorem 10, page 63.

In addition we have recalled the situation that occurs when the switching curve is linear. We have also, at the end, touched upon certain problems that arise when the equation is nonlinear, notably of the vander Pol type. This case, however, is so much more complicated than the linear case that no attempt has been made to give a complete solution.

ACKNOWLEDGEMENTS

The research on which this thesis is based was sponsored by the Office of Naval Research* in connection with a project centered at the Stevens Institute of Technology.

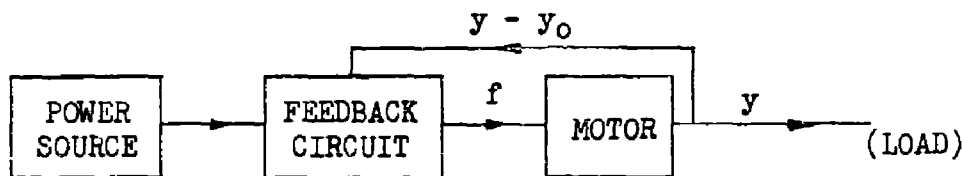
It owes much to Professor S. Lefschetz and Mr. R.R. Williamson, and the writer wishes to express his gratitude for their manifold assistance and support. Thanks are due also to Professor S.P. Diliberto for some valuable advice and to Professor L.L. Rauch for several brief but enlightening discussions.

Most of the figures in Section II were adapted from Flügge-Lotz (2).

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I. INTRODUCTION

A simple example will serve to illustrate the kind of physical situation in which the central problem of this paper arises. Consider, therefore, a servomechanism which consists of a motor, a source of power, and a feedback circuit. We shall suppose that the purpose of the servomechanism is to hold the output of the motor constant, say at the value y_0 , where to fix the ideas we shall suppose that the output of the motor is measured in terms of the angle y of the rotor with respect to some fixed reference



position. The feedback circuit is sensitive to errors in y , i.e., to the quantity $y - y_0$, and its object is, upon sensing such an error, to apply to the motor an input the effect of which will be to tend to nullify this error. The input to the motor is denoted by f , and will be supposed to depend on $x = y - y_0$ and the first derivative with respect to time of this quantity.

The differential equation for such a system can be taken to be^{*}

$$I \frac{d^2 y}{dt^2} + R \frac{dy}{dt} = K \cdot f \left(x, \frac{dx}{dt} \right)$$

or, equivalently,

$$I \frac{d^2 x}{dt^2} + R \frac{dx}{dt} = K \cdot f \left(x, \frac{dx}{dt} \right) \quad (1)$$

where I is a constant representing the moment of inertia of the rotor, R is a constant representing various sources of energy dissipation in the system, and K is the "torque constant." For a given motor (so

* See, e.g., James, Nichols, and Phillips: Theory of Servomechanisms (1947), p. 14.

that I , R , K are fixed) the crucial element in this equation is naturally the function f , which is determined by the design of the feedback circuit and by the strength of the power source.

The problem to be considered here arises when one puts a further assumption on f , namely the assumption that this function can take on only the two values $\pm\beta$, where β is a certain positive constant. This situation occurs when the feedback circuit acts simply as a switch (relay) which applies to the motor the full strength of the power source either directly ($+\beta$) or after inverting its polarity ($-\beta$). This scheme has the prima facie advantages that (1) the feedback circuit, since it no longer needs to yield a continuously varying output, can be vastly simplified; and (2) it would seem likely that, by always using the full strength of the power source, one should be able to smash any transient errors to zero more rapidly than by any other means. For the first reason, such servos are in fact extensively used; but it seems to be the opinion of many experts that the second reason is not sound, for such an intense, "bang-bang" servo is too crude to give a delicate response, and is prone to display several kinds of highly undesirable behavior: high-frequency, low-amplitude oscillations ("chattering"), medium-frequency oscillations of constant or increasing amplitude ("hunting"), and others. But naturally all this depends on the character of f , and there is no reason to deny the possibility that there may exist some function f or class of such functions which would avoid these unpleasant phenomena and, in fact, give excellent performance. It will be proved below that there do exist such functions and that there even exists a unique such function f which gives the best possible performance, where "best possible" has a certain natural and definite meaning. This will indeed be shown for the more general equation

$$m \frac{d^2x}{dt^2} + r \frac{dx}{dt} + cx = D \cdot f \left(x, \frac{dx}{dt} \right) \quad , \quad (2)$$

where m , r , c , and D are constants and $f \left(x, \frac{dx}{dt} \right) = \pm 1$. This equation can be put into one of the three simple forms

$$\frac{d^2 u}{d\tau^2} + 2b \frac{du}{d\tau} + u = \phi\left(u, \frac{du}{d\tau}\right), \quad \phi = \pm 1, \quad (2')$$

$$\frac{d^2 u}{d\tau^2} + b \frac{du}{d\tau} = \psi\left(u, \frac{du}{d\tau}\right), \quad \psi = \pm 1, \quad (2'')$$

$$\frac{d^2 u}{d\tau^2} + 2b \frac{du}{d\tau} - u = \theta\left(u, \frac{du}{d\tau}\right), \quad \theta = \pm 1, \quad (2''')$$

by choosing the proper units for time (t) and error (x), according as c is positive, zero, or negative. (The constants m and D are assumed to be different from zero; it is no restriction of the generality to suppose that they are both positive.) For instance, to get (2') from (2) when $c > 0$, put

$$x = \frac{D}{c} u, \quad t = \sqrt{\frac{m}{c}} \tau, \quad b = \frac{1}{2} \frac{rD}{\sqrt{cm^3}}$$

The problem which will be treated in this paper is:

In

$$\frac{d^2 x}{dt^2} + g\left(x, \frac{dx}{dt}\right) = \phi\left(x, \frac{dx}{dt}\right), \quad (3)$$

where $g(x,y)$ is a given function of class C^1 and $\phi(x,y)$ is a function which assumes only the values ± 1 , how should $\phi(x,y)$ be chosen so that the solution of (3) for any set of initial conditions (x_0, \dot{x}_0) ($\dot{x} = \frac{dx}{dt}$) reaches the state $x = 0, \dot{x} = 0$, and in fact reaches it in less time than for any other choice of $\phi(x,y)$?

(In terms of our mechanical example, this means: how should the feedback circuit be designed so that if the output suddenly undergoes a disturbance which results in a certain error and rate of change of error, then these two quantities are brought back to zero simultaneously and as rapidly as possible?)

A restricted form of the problem has been thoroughly studied by a group of people represented by I. Flügge-Lotz and K. Klotter (see the

bibliography); they treat the equation (2') with $0 < b < 1$ and assume that ϕ has the special form

$$\phi(x,y) = \text{sgn}(Kx + My)$$

where sgn is the function whose value is $+1$ for positive argument and -1 for negative argument, and K, M are constants. Their results will be outlined in II, where it will appear that in this case the problem as stated above is insoluble and that the problem of choosing the "best" ϕ , i.e., the "best" values for K and M , becomes one of avoiding as many undesirable phenomena as possible. The "best" ϕ so determined indeed depends for its efficacy on the assumption that the equation (2') gives an essentially incomplete description of the physical situation, that in fact "time lags" occur. D. McDonald (McDonald (1)) has discussed the problem for the equation (2'') with ϕ again general and has stated the correct result for this case, on the basis of a heuristic argument. Except for these, no results have been given for the problem stated above.

The first principal new result of this paper will be Theorem 1, which greatly restricts the class of functions $\phi(x,y)$ which one needs to consider in seeking a solution; then, on the basis of this theorem, the problem is solved for all linear $g(x,y)$. After this something is said about the nonlinear case, and the paper concludes with brief discussions of some distinct but closely related problems.

It should be remarked that the problem admits of various generalizations, for none of which significant results are known. For example, one can consider a higher order differential equation, say of order n ($n > 2$) and require that on any solution the quantities $x, \frac{dx}{dt}, \dots, \frac{d^k x}{dt^k}$ ($k < n$) should at some instant vanish simultaneously and in the shortest possible time; or one might consider systems of equations, each involving a different function of the type $\phi(x,y)$, for example:

$$\frac{d^2 x}{dt^2} + g_1\left(x, \frac{dx}{dt}, y, \frac{dy}{dt}\right) = \phi_1\left(x, \frac{dx}{dt}, y, \frac{dy}{dt}\right)$$

$$\frac{d^2 y}{dt^2} + g_2\left(x, \frac{dx}{dt}, y, \frac{dy}{dt}\right) = \phi_2\left(x, \frac{dx}{dt}, y, \frac{dy}{dt}\right) .$$

(This would represent the problem for a mechanical system with two degrees of freedom with coupling.) Problems of a different character arise when one remains with the equation (3) but supposes that the externally caused errors are not of the simple, square-wave type we have considered, but of some more intricate but statistically describable type. In view, however, of the difficulties involved in dealing with the simpler problem here discussed, the possibility of obtaining significant general results for the more complicated problems seems, at present, rather remote.

Terminology

The equation (3) is equivalent with the system

$$\frac{dx}{dt} = y$$

(4)

$$\frac{dy}{dt} = \phi(x,y) - g(x,y)$$

where $g(x,y) \in C^1$ and $\phi(x,y) = \pm 1$.

Suppose for the moment that $\phi(x,y) \equiv +1$. Then the system (4) has a unique solution through every point (x_0, y_0) of the plane; in other words, the family of curves defined by (4) with $\phi(x,y) \equiv +1$ covers the entire plane exactly once. This family of curves will be called the P-system (P for positive), its curves P-curves, and the arcs of its curves P-arcs.

In the same way, when $\phi(x,y) \equiv -1$, one gets another family of curves covering the plane; this is the N-system (N for negative) and N-curves and N-arcs are defined correspondingly.

A positive direction of motion (an orientation) on each P- or N-curve is automatically defined in terms of t .

Now if $\phi(x,y)$ is allowed to be as before, i.e., any single-valued function whose domain is the entire plane and whose range is confined to the values of ± 1 , the classical theory of differential equations does not provide a definition for the notion of a solution of (4). If $\phi(x,y)$ is simple enough it becomes clear how such a solution should be defined; but when $\phi(x,y)$ is general this is not so clear, and some care must be taken.

The definition which follows seems best. It is expressed mainly in geometrical terms, but the translation to analytical language is easy.

Suppose that $p_0 = (x_0, y_0)$ is the point from which the solution is sought, and suppose $\phi(x_0, y_0) = +1$. Then one of the three following mutually exclusive possibilities must be realized:

(i) There exists a P-arc beginning at p_0 of positive length along which $\phi(x, y) = +1$.

(ii) The condition (i) is not satisfied, but there exists an N-arc from p_0 along which $\phi(x, y) = -1$ (excluding p_0).

(iii) Neither (i) nor (ii) holds.

If (i), either there exists a first point* after p_0 on the P-curve from p_0 at which $\phi(x, y)$ changes sign, or there does not. If there does, the solution is defined to begin with the P-arc from p_0 to this point p_1 . If there does not, the solution from p_0 is defined to be that part of the P-curve through p_0 which follows p_0 (the P-semicurve from p_0).

If (ii), the preceding paragraph should be applied with N in place of P.

If (iii), no solution from p_0 is defined.

Cases (i) and (ii) thus lead either to a definition of the entire solution from p_0 , or to a definition of the solution up to some definite point $p_1 = (x_1, y_1)$. In the latter case, the above process should be repeated, with p_1 in place of p_0 , the letters P and N interchanged, the numbers ± 1 interchanged, and the phrase "excluding p_1 " added at the end of (i). This will lead to the same trichotomy: either the solution is not defined beyond p_1 , or it consists of a whole N- or P-semicurve beginning at p_1 , or it follows p_1 with a definite arc $p_1 p_2$. Then the whole process should be applied to p_2 (when this point occurs; otherwise there is nothing left to do); but now P, N, and ± 1 should be in their original places. This either accounts for the rest of the solution, or leads to a point p_3 like

* To be precise, let $x(t), y(t)$ represent the P-curve such that $x(0) = x_0, y(0) = y_0$, and let $\tau = \inf \{t \mid t > 0, \phi[x(t), y(t)] = -1\}$. Then the "first point" mentioned is $(x(\tau), y(\tau))$.

p_1 which should be treated like p_1 ; and so on. The whole curve obtained in this way is, by definition, the solution of (4) from p_0 .

If $\phi(x_0, y_0) = -1$, the solution from p_0 is defined correspondingly.

If each of the points p_n ($n = 0, 1, 2, \dots$) is assigned to the adjacent P-arc if $\phi(x_n, y_n) = +1$, the adjacent N-arc if $\phi(x_n, y_n) = -1$,^{*} then the solution of (4) from p_0 consists of a countable (possibly finite or even vacuous) well-ordered sequence of alternating P- and N-arcs such that the initial point of the first arc is p_0 , the terminal point of each arc is the initial point of the next, and $\phi(x, y) = +1$ on the P-arcs, -1 on the N-arcs.

The solution of (4) from a point p , if it exists, is unique. This follows at once from the definition and the fact that P- and N-curves are unique.

It is also easy to see that if Δ is the solution of (4) from p , and p' is any point on Δ , then the solution from p' is that part of Δ which follows p .

From these facts it follows in turn that a solution cannot intersect itself at a point p unless it is periodic beyond p . (Here a solution can be "periodic beyond a point" without being completely periodic, despite the uniqueness, because our solutions are defined only unilaterally.)

A point on a solution which is the terminal point of a P-arc and the initial point of an N-arc will be called a PN-corner. NP corners are defined analogously.

II. LINEAR SWITCHING

Linear switching occurs when $\phi(x, y) = \text{sgn}(Kx + My)$, i.e., when $\phi(x, y) = 1$ in one of the half-planes determined by the line $Kx + My = 0$ and -1 in the other. This case for the equation

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + x = \phi\left(x, \frac{dx}{dt}\right) \quad (0 < b < 1) \quad (2')$$

^{*} If, e.g., $\phi(x_n, y_n) = +1$ while the two adjacent arcs are both N-arcs, p_n is to be regarded as a degenerate P-arc.

has been thoroughly discussed in several papers (Flügge-Lotz (1),(2); Flügge-Lotz and Klotter (1)). The summary of their results given in this section (based on Flügge-Lotz (2), Chap. 4) will serve the double purpose of showing how much can be done with such a $\phi(x,y)$ and of displaying some of the unwelcome phenomena that can occur in such problems.

In this case (4) becomes

$$\begin{aligned}\frac{dx}{dt} &= y \\ \frac{dy}{dt} &= -x - 2by + \operatorname{sgn}(Kx + My) \quad .\end{aligned}\tag{5}$$

The constant b is taken as fixed, and the focus of attention is the pair of constants K, M . We shall suppose that they are both different from zero; what happens when either of them vanishes is essentially the same as what happens in one of the other cases. There will then be four cases to consider, as tabulated:

	$K > 0$	$K < 0$
$M > 0$	I	III
$M < 0$	II	IV

As is well known, the equations for the P- and N-curves belonging to (5) can be explicitly computed; they may be expressed in the form

$$x(t) = e^{-bt} (Ae^{iat} + Be^{-iat}) + 1 \tag{6}$$

$$y(t) = -e^{-bt} [(b - ai)Ae^{iat} + (b + ai)Be^{-iat}]$$

where $\alpha = +\sqrt{1 - b^2}$ and

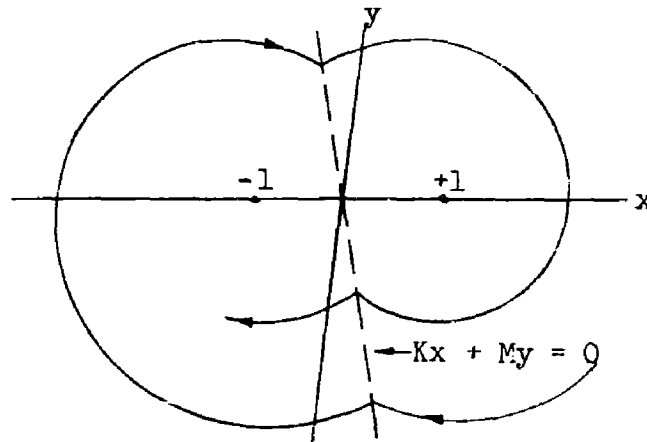
$$A = \frac{1}{2ai} \left\{ y(0) + (b + ai) [x(0) + 1] \right\}, \quad B = \bar{A} \quad . \tag{7}$$

Wherever an equivocal sign occurs, the upper sign pertains to the P-system, the lower to the N-system. These equations represent spirals spiralling into the foci $(\pm 1, 0)$.

A solution of (5) can thus be obtained explicitly in terms of a sequence of formula pairs (6), defined on successive intervals of time, each representing that part of the solution between two successive zeros of $Kx + My$.

Case I. In this case the "switching line" $Kx + My = 0$ passes through the second and fourth quadrants. To the right of it, $\phi(x,y) = +1$; to the left, -1 . Thus a solution consists of a sequence of arcs of spirals, each with its focus on its own side of the line $Kx + My = 0$ and its ends on this line.

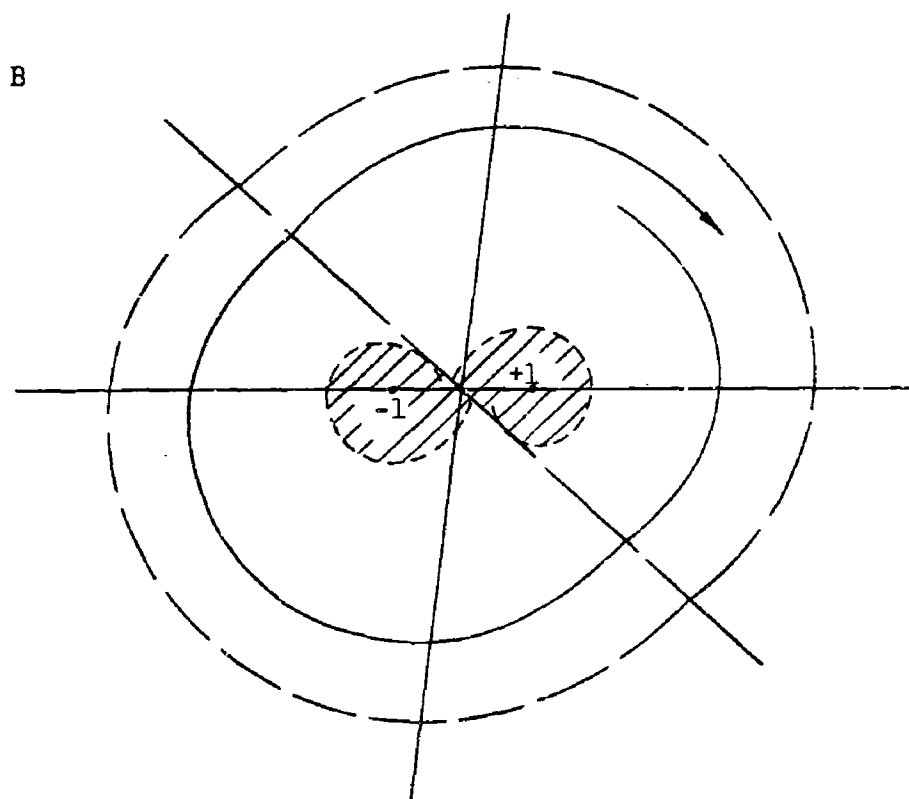
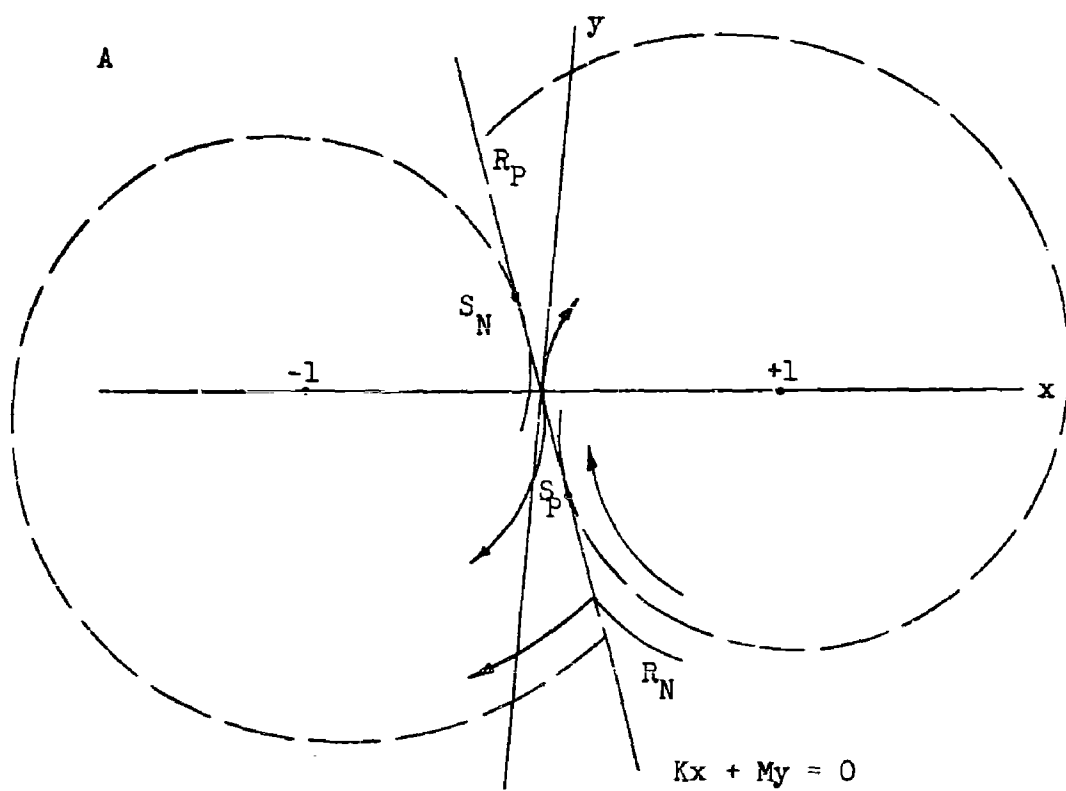
The condition for the existence of a periodic solution is that there should exist (say) a P-arc of the type described whose end-points are



CASE I

equidistant from the origin; for then, by symmetry, there exists an N-arc on the other side of the switching line joining the same two points, and these two arcs together form the periodic solution. We shall see below that a periodic solution can occur in our case.

Let S_P, S_N be the points on $Kx + My = 0$ where a P- and N-curve respectively are tangent; let R_P and R_N be the last intersections preceding S_P and S_N with the switching line of the P- and N-curves through these points. S_P and S_N are symmetric with respect to the origin, as are R_P and R_N . Suppose that b, K , and M are such that R_N is outside the closed segment $S_P S_N$; then the situation is as shown in Figure A (next page). A solution starting sufficiently near the segment $S_P S_N$ will move away from the line in one direction or the other -- according to the

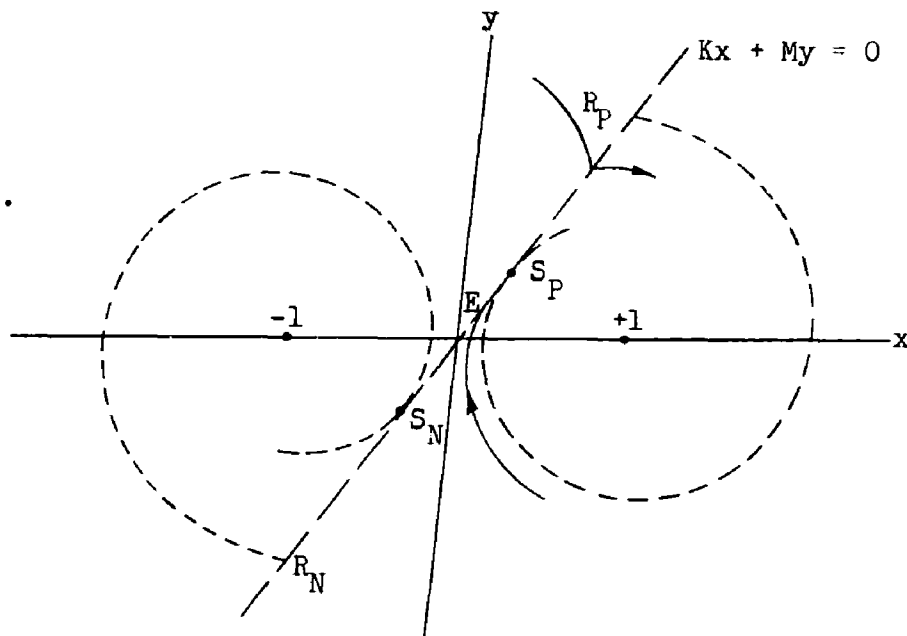


side of the line on which its initial point lies -- and never return to the switching line at all. Since solutions can only start near $S_P S_N$, but cannot cross it, the points of this segment are called start points. (Strictly speaking, $\phi(x,y)$ is not defined in this segment, so that one cannot speak of a solution starting on it.) Also, since any solution starting in this manner does not return to the switching line but merely spirals down to one focus or the other, such points are also rest points; the control represented by the function ϕ is at rest on the solution from such a point. It is easy to see that the points of $R_N S_P$ and $S_N R_P$ are also rest points, but not start points. In general, irrespective of the values of b , K , and M (for Case I), the segments $S_P R_P$ and $R_N S_N$ consist of rest points and the segment $S_P S_N$ consists of start points, as one can easily convince himself.

In the case illustrated in Figure A no periodic solution can exist; for it can be shown that every P-arc which lies to the right of the line $Kx + My = 0$ and has its ends on this line also has the property that its terminal point is nearer the origin than its initial point; thus the condition for a periodic solution can never be satisfied.

If, however, R_N and R_P are on the segment $S_P S_N$ (Figure B), then there exists a periodic solution. This may be seen as follows: the particular P-arc $R_P S_P$ begins nearer the origin than it ends, by assumption; but P-arcs which begin sufficiently far from the origin on the switching line have the reverse property; therefore, by continuity, some intermediate P-arc of this type must begin and end at the same distance from the origin; and this is exactly the condition for a periodic solution. Extended analysis bears this out, and shows that the periodic solution is unique and orbitally stable. All solutions beginning outside the periodic solution spiral onto it, and those solutions which begin inside the periodic solution but outside the shaded area (which represents rest points) also spiral onto it. When there is no periodic solution, as in Figure A, all solutions have finitely many corners and then spiral down with no further corners to one of the foci $(\pm 1, 0)$. Obviously, none of these kinds of behavior is welcome in terms of the problem at hand.

Case II. This case differs from the first in that the switching line $Kx + My = 0$ now passes through the first and third quadrants. The arcs which occur in solutions are as in Case I. No periodic solutions occur in this case. Let R_P, R_N, S_P, S_N be defined as before; then the points $R_P, S_P, 0, S_N, R_N$ lie on the switching line in this order. The intervals $R_P S_P$ and $S_N R_N$ are easily seen to consist of rest points; but on the interval $S_P S_N$ a new phenomenon occurs. Consider any solution which reaches this interval, say at the point E. What does the solution do at

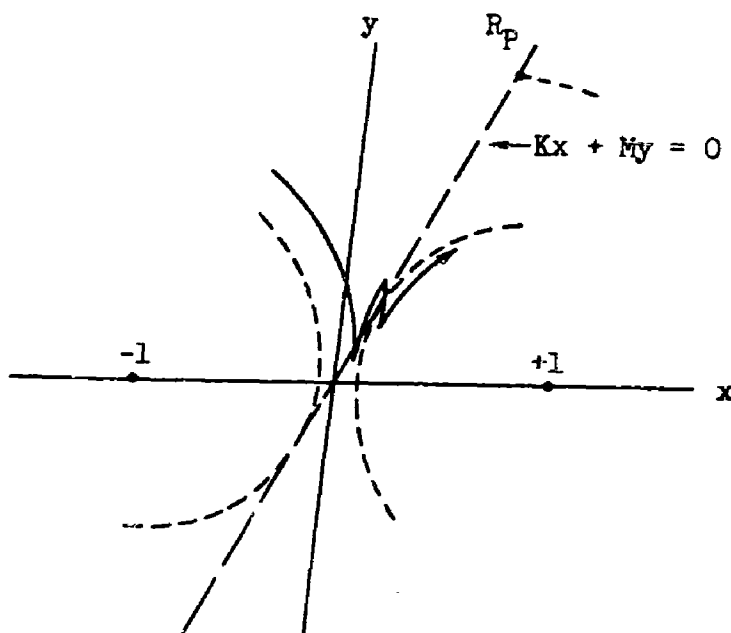


CASE II

this point? It should have a PN-corner at E, for it has reached a point where ϕ changes sign; but the N-curve from E goes back into the same half-plane from which the solution entered E, and on this side a solution can contain only P-arcs; on the other hand, the solution certainly cannot follow the P-curve through E beyond this point. Thus the solution is not defined beyond E; it ends at E. For this reason, such a point is called an end point. In a manner of speaking, end points are inverted start points and, like start points (but unlike rest points) can occur on the switching line only.

Any solution starting outside the region just considered spirals in toward the origin until it reaches a point on the interval $R_N R_P$, beyond which its behavior is determined by the above considerations.

In practice, due to mechanical traits of the physical system involved which prevent it from obeying our idealized hypotheses exactly, there arises a time lag, this means that a solution meeting the line $Kx + My = 0$ actually proceeds for some distance beyond it before it has the corresponding corner. In Case I such a time lag, provided that it is not too large, does not affect the essential behavior of the system; such a system might therefore be said to be "structurally stable with respect to time lags." But in Case II the presence of a time lag does make a difference; for consider a solution entering an end point; because of the time lag, it no longer ends there, but proceeds for a certain distance beyond and then has a corner, where a solution is still defined. From this corner it crosses the switching line in the reverse direction, moves for a short distance beyond, has another corner, and so on. The successive intersections of such a solution with the switching line move away from the origin, so that sooner or later one of the corners lies in the set of rest points, and from this corner the solution proceeds to spiral down, without further corners, to the corresponding focus. (See the picture below.)

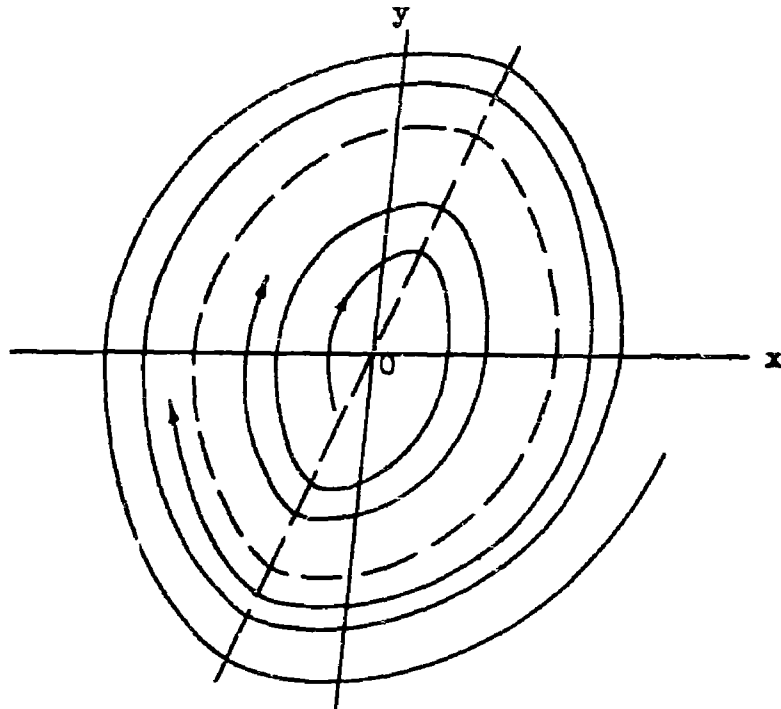


CASE II WITH TIME LAG

The situation in Case II can thus be summarized as follows: In the absence of time lags, every solution either terminates in an end point on $S_P S_N$,

or it eventually spirals down to one of the two foci $(\pm 1, 0)$. In the presence of a time lag, all solutions behave in the latter way. Thus Case II is also unfavorable from our point of view.

Case III. In this case the switching line lies as in Case II, but the arcs which occur in solutions now belong to spirals about the focus on the side of the line opposite from the arc itself. In other words, P-arcs occur on the left, N-arcs on the right. In this case a stable periodic solution always exists, and it dominates the whole situation, for all other solutions spiral onto it.



CASE III

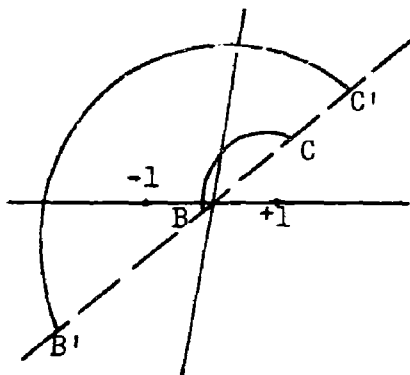
That a periodic solution exists can be seen as follows: consider the two arcs BC and $B'C'$, where B lies very near the origin; then

$$\overline{OC} - \overline{OB} > 0.$$

But, by the character of the spirals, if B' is far enough out

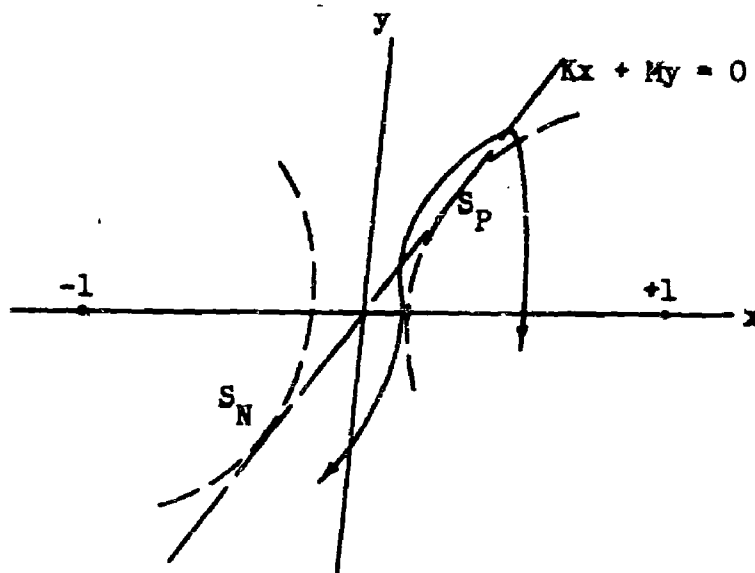
$$\overline{OC'} - \overline{OB'} < 0.$$

Thus, by continuity, there must be an intermediate arc $B''C''$ such that



$\overline{OC}'' - \overline{OB}'' = 0$; but this, as pointed out earlier (p. 9), is exactly the condition for the occurrence of a periodic solution. An extended discussion of such periodic solutions can be found in Bilharz (1); quantitative information about the particular periodic solutions arising here, for varying values of the parameters, are given in Flügge-Lotz (2).

The behavior of the solutions within the periodic solution is simple enough. It is clear that, if S_P and S_N are the points of tangency defined as before, the segment $S_P S_N$ consists of starting points. A solution starting from such a point (in either direction) simply spirals out to the periodic solution; and since the totality of solutions obtained in this way covers the interior of the region bounded by the periodic solution, there are no other solutions to consider. (See the picture below.)

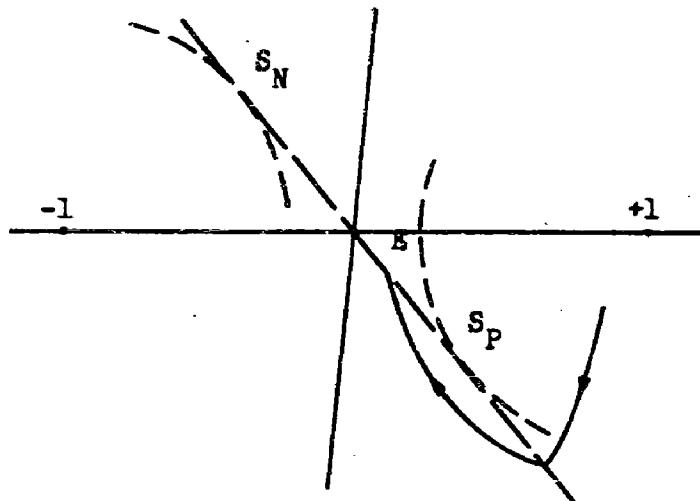


CASE III NEAR THE ORIGIN

Thus all solutions spiral onto the periodic solution and this, from the point of view we have adopted, is also unfavorable. It may be seen that in this case, as in Case I, we have "structural stability with respect to time lags"; i.e., the presence of a small time lag would not change things essentially.

Case IV. In this case the switching line is as in Case I, the arcs of solutions as in Case III. It may be shown that no periodic solution can exist in this case; in fact, every arc with its ends on the switching

line has its terminal point nearer the origin than its initial point, and the condition for a periodic solution cannot be satisfied.

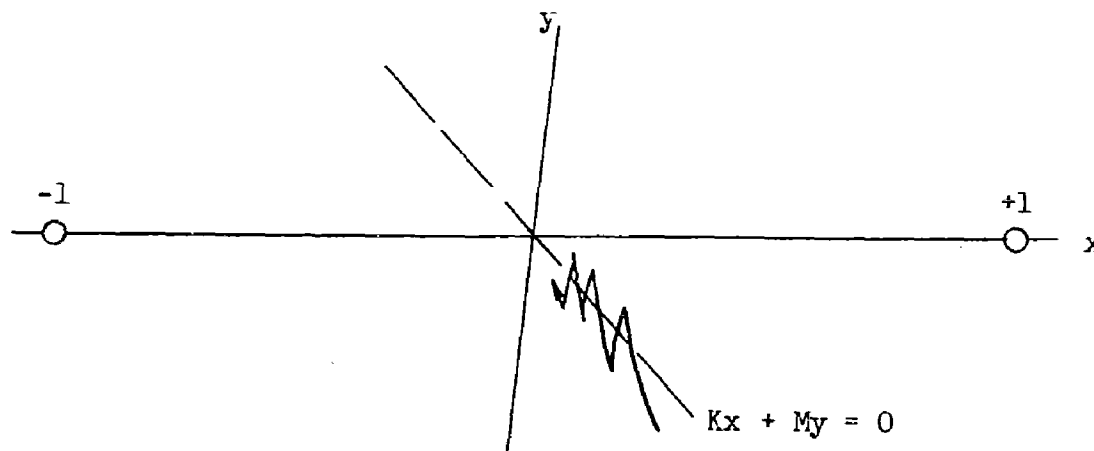


CASE IV WITHOUT TIME LAG

In this case the segment $S_P S_N$ consists of end points, and by tracing the solutions which end on it backwards one can see that these cover the entire plane; thus in this case all solutions end on the segment $S_P S_N$ of the switching line.

But here again, as in Case II, the presence of a time lag makes a difference. The time lag makes no difference of importance until the solution in question reaches $S_P S_N$; then, instead of ending, the solution proceeds for some small distance beyond the switching line, has a corner, recrosses the switching line, has another corner, and so on. It may be seen that the successive points of crossing obtained in this way have the property that each is closer to the origin than its predecessor, until one of them lies on the other side of the origin. After this has happened the solution oscillates around the origin in a more or less irregular way, but with a high mean frequency and small mean amplitude. This is the most favorable of the possibilities so far considered, for every solution moves into the origin with the passing of time, and this irrespective of the magnitude of the constants b , K , and M . But the manner in which it does so is unsatisfactory, for it involves a rapid fluctuation in the sign of ϕ ("chattering") which, in general, continues indefinitely.

Thus, at best, linear switching leads to solution behavior which, both qualitatively and (as we shall see) quantitatively, is far from perfect; it will be shown later (Theorem 7) that all its defects can be avoided by taking a different kind of ϕ .



CASE IV WITH TIME LAG

III. THE MINIMAL THEORY; GENERAL CONSIDERATIONS

We now return to the problem stated at the bottom of page 3. If the second order equation (3) is replaced by the equivalent first-order system (4), the problem may be described as that of finding a function $\phi(x,y)$ such that for any point p in the x,y -plane the solution from p of

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \phi(x,y) - g(x,y) \quad [\phi = \pm 1, g \in C^1] \quad (4)$$

has the following properties:

- (i) it passes through the origin ($x = y = 0$);
- (ii) the length of time necessary to move from p to the origin along the solution from p is minimal with respect to ϕ ; i.e., no other ϕ could make this time shorter.

(The parametrization of a solution in terms of t is naturally induced in the obvious way by the known parametrization in terms of t of its component arcs; thus the "time" required to move from one point on a solution to another which follows it is a well-defined quantity.)

The object of this section is to study the problem in this general form. It will first be shown how the problem is equivalent to one of a conceptually simpler character, and then (in Theorem 1) it will be shown that this problem in turn can be greatly simplified.

It has been pointed out that a solution of (4) consists of a sequence of alternating P- and N-arcs, the initial point of the first being the initial point of the solution, and the terminal point of each being the initial point of the next. Our point of view will be to consider, for an arbitrary point p in the plane, the class of all curves of this kind which begin at p and pass through the origin; and our purpose will be to find in this class a curve along which the time necessary to reach the origin is shortest. Thus we make the following definitions (the function $g(x,y)$, and therefore the P- and N-systems, being fixed):

A path from the point p is a finite or countable, well-ordered sequence of alternating P- and N-arcs such that:

- 1) The sum of the lengths* of the arcs is finite ($= \tau$).
- 2) The initial point of the first arc is p .
- 3) The terminal point of each arc is the initial point of the next.

4) If there are finitely many arcs, the terminal point of the last arc is the origin; if there are infinitely many, then

$$x(t) \rightarrow 0 \text{ and } y(t) \rightarrow 0 \text{ as } t \rightarrow \tau,$$

the curve composed of the arcs being parametrized in the obvious way in terms of the parametrization of the component arcs.

- 5) No two of the arcs intersect.

(In order to avoid a conflict between 3) and 5), we regard each arc as containing its initial point but not its terminal point; this convention will have no effect on the time-length characteristics of paths, and therefore does not really restrict the generality of what follows.)

* "Length," "longer," and similar expressions should be understood to refer to time, not geometric length, here and throughout what follows.

A path from p can therefore almost be described as a curve which could occur as that part of a solution from p (for some ϕ) which connects p with the origin. "Almost," because 5) need not hold for a solution of (4). This point will be cleared up presently.

A path from p which is not longer than any other path from p will be called a minimal path from p .

In order to solve the problem stated on page 17 it is sufficient to find a unique minimal path from each point p .

Namely, one needs only to define $\phi(x,y) = +1$ on P-arcs which occur in the minimal paths, and $\phi(x,y) = -1$ on the N-arcs which occur in the minimal paths. ($\phi(0,0)$ is to be left undefined, or it can be given either value.) Such a $\phi(x,y)$ automatically yields the minimal paths as solutions, and the minimal path from a point is, by definition, the shortest possible solution connecting p with the origin. Two things must be verified: (1) that this method defines $\phi(x,y)$ uniquely at every point except the origin, and (2) that nothing is lost by leaving out of consideration those possible solutions for which 5) fails.

To verify (1), observe first that every point p must lie on at least one minimal path, namely the minimal path which begins at p . Thus $\phi(x,y)$ is defined everywhere. If there were some point at which it failed to be unique, then this point p would need to lie both on an N-arc belonging to one minimal path Δ_a (from the point a) and on a P-arc belonging to another minimal path Δ_b (from b). Denote those parts of Δ_a and Δ_b which lie between p and the origin by Δ'_a and Δ'_b respectively; then their time lengths $\tau(\Delta'_a)$ and $\tau(\Delta'_b)$ stand in some relation to each other, say $\tau(\Delta'_a) \leq \tau(\Delta'_b)$. Then $\tau(\Delta_b - \Delta'_b + \Delta'_a) \leq \tau(\Delta_b)$. $\Delta_b - \Delta'_b + \Delta'_a$ may not be a true path (for it may cross itself), but a true path $\tilde{\Delta}_b$ may be obtained from it by cutting out whatever closed loops or retracings it may contain; and obviously $\tau(\tilde{\Delta}_b) \leq \tau(\Delta_b)$, which contradicts the assumption that Δ_b was the unique minimal path from b .

To check (2), it will suffice to show that any curve Δ which might occur as a solution connecting p with the origin and failing to satisfy 5) can be replaced by another such curve, at least as short, which satisfies 5) and is therefore a path. Let Δ be as described, and let Δ' be the

path obtained from Δ by cutting off all of Δ beyond the first intersection of this curve with the origin. Δ' contains the origin just once, and ends there; since Δ' is not longer than Δ , it is enough to show that Δ' satisfies 5). Suppose the contrary; if Δ' intersects itself at a point q beyond which, by page 7, it is therefore periodic, it follows that Δ' contains the origin at least twice: once following the second passage through q (for Δ' ends at the origin) and therefore on the period connecting the first passage of Δ' through q with the second. This is a contradiction.

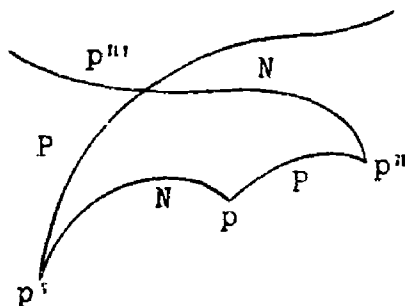
The problem with which we shall actually be concerned is therefore that of finding a unique minimal path from any point p , given the function $g(x,y)$.

A path will be called canonical if it contains no NP-corners (see p. 7) above the x-axis and no PN-corners below.

Theorem 1. Given any path Δ from p which is not canonical, one can find a canonical path from p which is shorter (in terms of time) than Δ .

Proof. (In saying that a corner lies above or below the x-axis, we mean that the arcs adjacent to the corner are, for values of t sufficiently near the value corresponding to the corner itself, above or below the x-axis respectively; the corner itself, regarded as a point, may thus lie on the axis.)

The idea of the proof is simple: given, say, a path with the NP-corner p above the x-axis, one denotes by p' either the last corner of the path preceding p or the last intersection preceding p of the path with the x-axis (whichever is nearer p), and denotes by p'' the corresponding point with "following" in place of "preceding" and "first" in place of "last." One then draws the P-curve forward from p' and the N-curve backward from p'' , thereby obtaining a four-sided figure as shown. If one now modifies the given path by replacing $p'pp''$ by $p'p''p$,



the NP-corner p is removed, no other such corner is introduced, and the path is shortened. To see this last fact, note that, by (4),

$$r(p'pp'') = \int_{p'pp''} \frac{dx}{y}, \quad r(p'p'''p'') = \int_{p'p'''p''} \frac{dx}{y}$$

(These two integrals must converge, for the quantities $r(p'pp'')$ and $r(p'p'''p'')$ are obviously finite.) However, y is greater (for a given value of x) on $p'p'''p''$ than on $p'pp''$; therefore the second integral is smaller than the first, as was claimed. Thus if one applies this process to every NP-corner above the x -axis, and the corresponding process to every PN-corner below the axis, one obtains a canonical path shorter than the given one.

Two things must be proved: (1) that it is always possible to construct the "quadrilateral" of the type shown; and (2) that the process described does not produce any self-crossings, so that a true path is in fact obtained.

Let p , p' , and p'' be as described above; if the initial point of a path is regarded as a corner, p' always exists, and p'' always exists since the path goes to the origin. It will be shown first that the P-semicurve Π beginning at p' passes over $p'pp''$ and crosses the vertical line through p'' . That Π moves to the right as long as it remains above the axis follows from the first equation in (4). Suppose that Π is parametrized by t in such a way that $t = 0$ gives p' ; then Π has one of the following two properties:

(i) Π goes arbitrarily close to the x -axis as $t \rightarrow \infty$ (i.e., it either crosses the x -axis for some $t > 0$ or $\lim_{t \rightarrow \infty} \inf y(t) [\text{on } \Pi] = 0$).

(ii) Π goes to infinity in the sense that

$$\sup \{x | (x, y) \in \Pi\} = +\infty$$

For assume that (i) is false; then there exists a number $\epsilon > 0$ and a value $t_0 > 0$ of t such that for $t > t_0$, $y(t) [\text{on } \Pi] > \epsilon$. Since $\frac{dx}{dt} = y$, this gives

$$x(t) = x(t_0) + \int_{t_0}^t y(\tau) d\tau > x(t_0) + \epsilon(t - t_0) \rightarrow \infty \text{ as } t \rightarrow \infty$$

so that (ii) holds.

Since, as t increases, Π moves steadily to the right, it is clear what Π must do; it must either move off to infinity as in (ii), or cross the x -axis at some point, or tend to some point on the x -axis as $t \rightarrow \infty$.

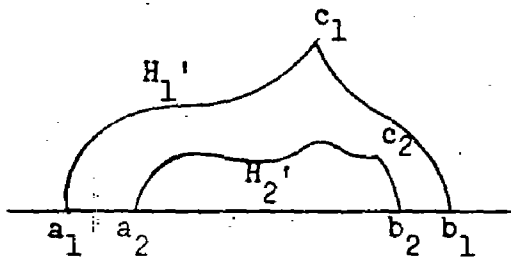
Π starts off from p' above $p'p''$, for in the upper half plane the P -curve through a point always has a greater slope there than the N -curve through that point, and even if p' lies on the x -axis (whereupon the two slopes are "equal" --- both infinite) the radius of curvature of the P -curve at p' is greater than that of the N -curve. (All this follows from (4); in particular, the fact that we can talk about radii of curvature follows from the fact that $g(x,y) \in C^1$, so that $x(t)$ and $y(t)$ --- the functions defining " Π " --- have continuous second derivatives.) Π cannot cross the N -arc $p'p$, by what was just said about slopes; it cannot cross the P -arc pp'' , for P -curves are unique; and it cannot tend as $t \rightarrow \infty$ to either of the points p or p'' (one or both of which may be on the x -axis) for this would imply that the point concerned would be a singular point of the P -system, which would in turn belie the fact that both points belong to the ordinary finite P -arc pp'' . Thus all that was claimed for Π is true.

The corresponding argument can be applied to N , the N -semicurve obtained by following the N -curve through p'' backwards, and it turns out that it too lies above $p'p''$ and, in particular, crosses the vertical line through p' . Thus Π and N must intersect at least once. That they intersect only once may be seen in several ways, the simplest of which perhaps is to observe that if they intersected twice (with no other intersections between), one of the intersections would involve a crossing with the wrong inequality between the slopes. Thus we obtain the unique intersection p''' and the "quadrilateral" sought.

This proves (1) on page 21. To prove (2), we note first that in the process just discussed (and its complement for the lower half-plane) the upper and lower half-planes are treated separately, so that in looking for possible self-crossings introduced by this process we need only consider

(say) what happens to those parts of the original path which lay above the x-axis. Let H_1 and H_2 , then, be any two parts of the original path, each contained between successive points where the path crosses the x-axis and in the upper half-plane. It is clear that the process for removing NP-corners cannot introduce self-crossings in either of the separate pieces H_1 or H_2 ; hence it is only necessary to show that the process can not cause H_1 to cross H_2 . Since H_1 and H_2 do not cross, their ends a_1, b_1, a_2, b_2 on the x-axis must lie in one of the following orders: (1) a_1, b_1, a_2, b_2 ; (2) a_1, a_2, b_2, b_1 ; (3) a_2, a_1, b_1, b_2 ; (4) a_2, b_2, a_1, b_1 . (1) is essentially the same as (4), and (2) as (3), so we consider only (1) and (2).

Our process has the property that it leaves the points where the given path crosses the x-axis unchanged; no such points are removed, and none are introduced. Therefore the curves H_1' and H_2' belonging to the final path and obtained from H_1 and H_2 by the process have the same ends on the x-axis as before. Now in case (1) there is nothing further to say, for H_1' lies entirely over the interval (a_i, b_i) ($i = 1, 2$); since these intervals are disjoint, H_1' and H_2' cannot intersect. In case (2), H_1' and



H_2' must be as shown; each H_i' consists of a P-arc followed by an N-arc (one of which might be vacuous). Suppose they intersected; say the intersection occurred on the arc a_2c_2 . Then, since a_2c_2 cannot intersect a_1c_1 (both of them being P-arcs), a_2c_2 must inter-

sect c_1b_1 ; by the previous argument about slopes, these two arcs can only intersect once; therefore c_2b_2 must intersect c_1b_1 , since b_2 lies between a_1 and b_1 . But this is impossible, for these are both N-arcs. The same line of reasoning applies if c_2b_2 is assumed to intersect a_1c_1 . Thus H_1' and H_2' are disjoint, as claimed; and this completes the proof of Theorem 1.

Corollary. In seeking a minimal path from a point it is only necessary to consider canonical paths from that point.

For it follows directly from Theorem 1 that a path which is minimal with respect to the class of all canonical paths is also minimal with respect to the class of all paths. From this point on it will therefore be tacitly assumed that all paths mentioned are canonical.

If $g(x,y)$ has the particular property that $g(-x,-y) = -g(x,y)$, there is more one can say. For then, if in the equations

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \phi(x,y) - g(x,y) \quad (4)$$

we make the substitutions $x = -X$, $y = -Y$, $\phi(x,y) = -\psi(X,Y)$, we obtain

$$\frac{dX}{dt} = Y, \quad \frac{dY}{dt} = \psi(X,Y) - g(X,Y)$$

i.e., equations of exactly the form (4). This means that if p and q are two points symmetrical with respect to the origin, then whatever can be said about the P- and N-curves at p can be said about the N- and P-curves at q. (E.g., if it can be proved that a minimal path from p must begin with a P-arc, it follows at once that a minimal path from q must begin with an N-arc.)

Since $g(x,y)$ always has the property mentioned when it is linear (homogeneous), and $g(x,y)$ will be of this type in most of what follows, this observation will find extensive use. Any result obtained from another by an appeal to it will be said to have been obtained by symmetry.

IV. THE MINIMAL THEORY: $g(x,y) = by$

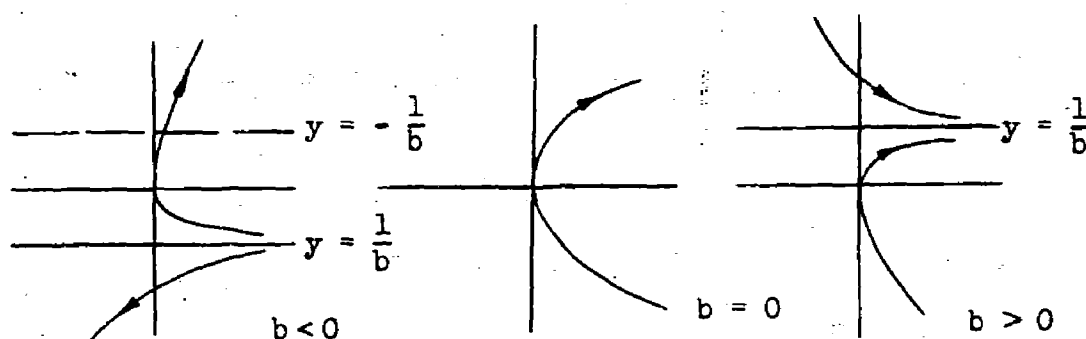
This section begins the systematic study of the problem discussed in III for the important case that $g(x,y)$ is linear. By page 3, it will suffice to suppose that $g(x,y)$ has one of the three forms by , $x + by$, $-x + by$, where in each case b is an arbitrary real constant. In this section we study the first of these; it arises from the physical example given at the very outset and, of course, in many other ways. The correct solution of the problem for $b \geq 0$ has been previously stated (McDonald (1)), but without a convincing argument.

The character of the P- and N-systems associated with the equations

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \phi(x, y) - by$$

depends, of course, on the value of b ; but this much can be said: since the left members (disregarding ϕ) depend only on y , all the P-curves can be obtained from any one of them by translation along the x -axis. The same, of course, can be said for the N-curves, which are in fact obtained from the P-curves by reflection in the origin.

The P-curves determined by the origin and (for $b \neq 0$) the point $(0, 2/b)$ for representative values of b are sketched below; that the solutions have the qualitative properties involved can be easily verified.



When $b < 0$, something strange (but not unexpected) occurs: there exist points from which there are no paths. More precisely, a path from the point (x_0, y_0) exists if and only if $|y_0| < -b^{-1}$. For suppose $y_0 < b^{-1} < 0$; then

$$\left. \frac{dy}{dt} \right|_{y=y_0} = 1 - by_0 < 1 - 1 = 0.$$

This means that once any curve, P- or N-, is below the line $y = b^{-1}$ it stays below; thus no path from such a point could cross this line, as it would need to do to reach the origin. Similarly for any point above the line $y = -b^{-1} > 0$. That paths exist from any point between the two lines will be seen below.

When $b \geq 0$, a path can be found from any point in the plane.

We shall denote by Γ that part of the P-curve through the origin which lies below $y = 0$, and by Γ^- its reflection in the origin. Γ^- is therefore that part of the N-curve through the origin which lies above $y = 0$.

Theorem 2. ($g(x,y) = by$.) Let $C = \Gamma^- + O + \Gamma$. This is a simple curve which divides the plane into an upper and a lower part. The unique minimal path from any point p above C (and below $y = -b^{-1}$, if $b < 0$) is obtained by following the N-curve from p until it reaches $C(\Gamma)$ and then following Γ into the origin. If p lies below C (and above $y = b^{-1}$, if $b < 0$) the unique minimal path is given by following the P-curve from p to $C(\Gamma^-)$ and then following Γ^- into the origin. (The solution of the original problem is to take $\phi = -1$ above C and on Γ ; $+1$ below C and on Γ^- .)

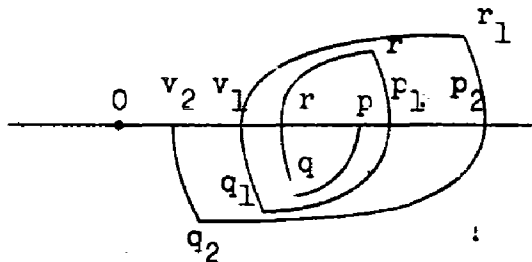
Proof. Note first that if pq is an N-arc with p on the x-axis and qr is the P-arc from q back to the axis, then $\tau(pqr)$, the total time length of this pair of arcs, is a monotone increasing function of the distance between p and r , or equivalently of the area bounded by pqr and the segment pr of the axis; for $\tau(pq)$ and $\tau(qr)$ are both increasing functions of $-y_q$, y_q being the ordinate of q ; but this in turn is such a function of the two quantities mentioned.

We shall denote that path from p which, according to the theorem, is minimal by $\tilde{\Delta}_p$. The rest of the proof will be broken up into several parts.

A. If p lies on the positive half of the x-axis, $\tilde{\Delta}_p$ is the unique minimal path from p .

Let Δ_p be any path from p other than $\tilde{\Delta}_p$. By page 24, Δ_p has no PN-corners below the axis or NP-corners above. It therefore starts out from p with an N-arc pq , which we may (see below) assume to be of positive length; q therefore lies below the axis. From q the path follows a P-arc qr which crosses the axis. (If r were the origin, we would have $\tilde{\Delta}_p$; and if r lay on or below the x-axis elsewhere, it would be a corner of the excluded type.) Let the intersection of qr with the x-axis

be v . The N-arc of Δ_p from r , say rq_1 , again crosses the axis or ends at the origin, for similar reasons; and so on. By continuing this kind of reasoning, and recalling that Δ_p does not cross itself, one discovers that Δ_p must be of the form shown. (To see what happens when the initial arc of Δ_p is a P-arc, think of v in place of p .) In order



for this path to reach the origin.

sooner or later one of the points v_n must lie at or to the left of the origin.* If p_n is that intersection of Δ_p with the x-axis which immediately precedes a v_n for which this

happens, then $p_n q_n v_n$ is an arc-pair of the type discussed at the beginning of the proof, and q_n is clearly lower than the NP-corner of $\tilde{\Delta}_p$; hence $\tau(\Delta_p) > \tau(p_n q_n v_n) > \tau(\tilde{\Delta}_p)$, which shows that $\tilde{\Delta}_p$ is the unique minimal path from p .

B. If p lies below $y = 0$ and above Γ (and, if $b > 0$, above $y = -b^{-1}$) then the unique minimal path from p is $\tilde{\Delta}_p$.

Suppose that Δ_p were a path from p such that $\tau(\Delta_p) \leq \tau(\tilde{\Delta}_p)$. One can join p to the x-axis by following the N-curve through p backwards. If the point on $y = 0$ reached in this way is p' , then $pp' + \Delta_p$ is (after the elimination of any loops, etc., that it may contain) a certain path (not necessarily canonical) from p' , say $\Delta_{p'}$; then

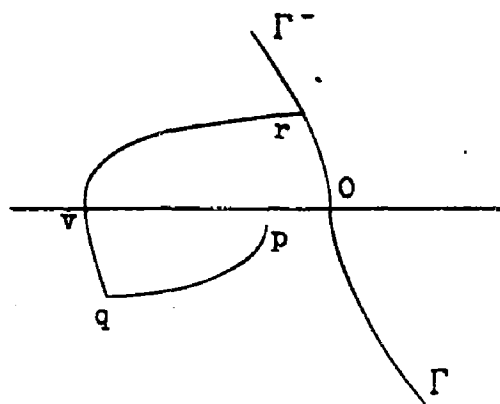
$$\begin{aligned} \tau(\Delta_{p'}) &\leq \tau(pp' + \Delta_p) = \tau(pp') + \tau(\Delta_p) \leq \tau(pp') + \tau(\tilde{\Delta}_p) \\ &= \tau(pp' + \tilde{\Delta}_p) = \tau(\tilde{\Delta}_{p'}) \end{aligned}$$

and this contradicts the fact (from A) that $\tilde{\Delta}_{p'}$ is the unique minimal path from p' .

C. If p lies below $y = 0$, to the left of Γ , and, if $b \neq 0$, above $y = -|b|^{-1}$, then $\tilde{\Delta}_p$ is the unique minimal path from p .

* They cannot merely approach 0 as a limit point, for this would imply the existence of infinitely many pieces like $p_n q_n v_n$; but each of these pieces would be longer than the innermost one, so that the length of the entire path would necessarily be infinite.

Consider any path other than $\tilde{\Delta}_p$ from p ; it starts out from p with an N-arc pq (necessarily lying entirely below $y = 0$) and from q follows a P-arc across the axis; but beyond this crossing, by the result obtained from A by symmetry, we may suppose that the path simply goes to



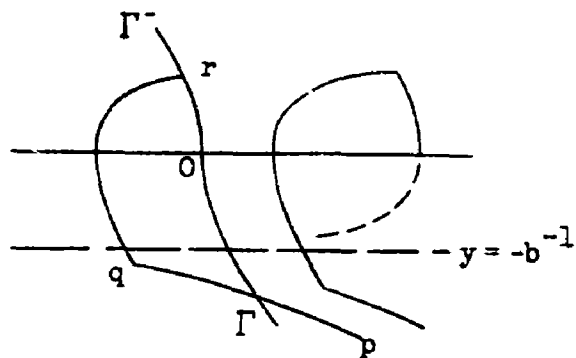
a point r on Γ^- and then follows Γ^- into the origin. Let the intersection of qr with the axis be v ; then $-x_v$, and therefore $\tau(vr0)$, is a strictly increasing function of $\tau(pq)$; likewise, since $\frac{dy}{dt} < 0$ on the N-curve from p , $\tau(qv)$ is such a function of $\tau(pq)$; altogether $\tau(\Delta_p)$ is thus an increasing function of $\tau(pq)$, and therefore

takes on its least value when $\tau(pq) = 0$; but this gives $\tilde{\Delta}_p$.

A, B, and C prove the theorem in the lower half-plane for every case but that in which $b > 0$ and p lies below $y = -b^{-1}$. The proof in B breaks down for this case because the N-semicurve ending at p does not reach the x-axis; the proof in C fails because $\frac{dy}{dt} > 0$ on the N-semicurve beginning at p . This bothersome case will be dispatched by proving a sequence of statements:

D₁. Any path from a point above Γ and below $y = -b^{-1}$ ($b > 0$) must begin with an N-arc which reaches Γ .

For if a path Δ_p began with an N-arc which fell short of Γ (if,



in particular, it were vacuous), the succeeding P-arc could not enter the origin, but could only cross the axis to the right of 0 and break off in a corner above. The N-arc from this point must again cross the axis, but it, as well as any P-arc following it, stays above the line $y = -b^{-1}$; thus such a path could not get to

the origin without crossing itself.

D_2 . If p is a point below the line $y = -b^{-1}$ ($b > 0$), and Δ_p is a path from p which starts along an N-arc pq , follows the P-curve from q to Γ^- and then follows Γ^- into the origin, then $\tau(\Delta_p)$ is a strictly increasing function of $\tau(pq)$.

Let the PN-corner (on Γ^-) of Δ_p be r , and write $\lambda = \tau(pq)$, $\mu = \tau(qr)$, $\sigma = \tau(rO)$; then it is to be proved that $\frac{d}{d\lambda}(\lambda + \mu + \sigma) > 0$. The best way to do this seems to be the following: it is easy to verify that the P- and N-systems are given by the equations

$$x(t) = A + Be^{-bt} + \frac{1}{b}t, \quad y(t) = -bBe^{-bt} + \frac{1}{b},$$

where

$$A = \left(y(0) + \frac{1}{b}\right)\frac{1}{b} + x(0), \quad B = -\frac{1}{b}\left(y(0) + \frac{1}{b}\right)$$

the upper signs giving the P-curves, the lower giving the N-curves. Using these formulae and the lengths of the three arcs of Δ_p , one can get expressions for the values of the coordinates of the two corners of this path, and of the origin regarded as the end of the third arc. Upon eliminating as much as possible from this system of equations, one obtains

$$\lambda - \mu + \sigma = bx_p + y_p = \text{constant}$$

and therefore, upon differentiating with respect to λ ,

$$\frac{d\mu}{d\lambda} = 1 + \frac{d\sigma}{d\lambda}$$

It is obvious that $\frac{d\sigma}{d\lambda} > 0$; therefore $\frac{d\mu}{d\lambda} > 0$, and the desired inequality follows at once.

Now it is easy to prove that for such a point $\tilde{\Delta}_p$ is the unique minimal path. In fact, if p is to the right of (above) Γ , a path Γ from p must, by D_1 , begin with an N-arc which goes at least far enough to meet Γ ; after meeting Γ the path may be supposed, as in

the proof of C, to be as described in D_2 . If p lies on the left of Γ , on the other hand, this may be supposed at once. But D_2 says that the shortest such path is obtained by taking the initial N-arc as short as possible. When p is to the right of Γ , this means taking q (the first corner) on Γ ; when to the left of Γ , taking the initial N-arc vacuous. In either case, the path so obtained is exactly $\tilde{\Delta}_p$.

A, B, C, and what has just been proved establish Theorem 2 for points in the open lower half-plane and on the positive half of the x-axis; the rest follows by symmetry.

V. THE MINIMAL THEORY: $g(x,y) = x + 2by$

This case corresponds to the equation (2) of page 3, and for it the equations (4) may be written

$$\frac{du}{dt} = v, \quad \frac{dv}{dt} = -1 - u - 2bv \quad (6)$$

We must distinguish two subcases: (i) $|b| < 1$, (ii) $|b| \geq 1$. These two subcases display essentially different kinds of qualitative behavior.

A. $|b| < 1$ (THE SPIRAL CASE)

When $|b| < 1$ it is well known that the P-system corresponding to (6) consists of spirals moving clockwise around the focus $(1,0)$. If $b > 0$ the spirals move in towards the focus (which is therefore stable), while if $b < 0$ the spirals move outwards, and the focus is unstable. If $b = 0$ the spirals degenerate to a family of circles and the focus becomes a center. The N-system may be obtained by translating the P-system two units to the left along the x-axis. It will be convenient to apply a linear transformation to the variables. Let $\alpha = +\sqrt{1-b^2}$; then the transformation is

$$x = u + bv, \quad y = \alpha v \quad (7)$$

This transformation leaves the axis of abscissae pointwise invariant; in fact, since it simply represents a change to an oblique coordinate system, it leaves all the essential properties of the original system undisturbed: PN-corners remain such, simple curves remain so, etc. Most important, the

time length of a given path is not changed by the transformation. It will therefore suffice to consider the x,y-plane only; any results obtained there may be applied in the u,v-plane by invoking the inverse of (7). Under (7) the original system (6) becomes

$$\begin{aligned}\frac{dx}{dt} &= -bx + ay + b \\ \frac{dy}{dt} &= -ax - by + a\end{aligned}\tag{8}$$

The solutions of (8) (i.e., the P- and N-curves) are given by:

$$\begin{aligned}x(t) &= e^{-bt}(Ae^{iat} + Be^{-iat}) + 1 \\ y(t) &= ie^{-bt}(Ae^{iat} - Be^{-iat})\end{aligned}\tag{9}$$

where

$$A = \frac{1}{2} [x(0) - iy(0) + 1] \quad , \quad B = \bar{A} \quad .\tag{10}$$

(Here and throughout the sequel the upper sign pertains to the P-system, the lower to the N-system.) The functions (9) represent ordinary logarithmic spirals or, if $b = 0$, circles.

The main result for the present case is embodied in Theorem 7; but since this is by far the most involved case to be considered, the proof will be broken up into a number of parts.

Lemma 1. The length of a P- or N-arc cut off from the corresponding kind of curve by two successive intersections with the x-axis is always π/a .

Proof. By (9) and (10), a given P- or N-curve intersects the x-axis when $Ae^{iat} = Be^{-iat} = \overline{Ae^{iat}}$ or, otherwise written, when $\text{Im} \{Ae^{iat}\} = 0$. Expanding this expression gives

$$\tan at = \frac{y(0)}{x(0) + 1}$$

For any but the singular solutions (the foci), the quantity on the right

is well defined or infinite; and since $\tan at$ takes on every such value at intervals of π/a , the statement follows.

Now a certain sequence of points on the x-axis is to be defined, as follows: First, take $\xi_0 = 0$. If one starts at $(\xi_0, 0) = (0, 0)$ and follows the P-curve from that point for the time $-\pi/a$ one reaches, by Lemma 1, a certain point $(\xi_1, 0)$ on the x-axis. If one then follows the N-curve from $(\xi_1, 0)$ for the same length of time, one reaches a point $(\xi_2, 0)$. Alternately following P- and N-arcs of length $-\pi/a$ in this way, one gets the sequence $(\xi_n, 0)$ which was to be defined.

$$\text{Lemma 2. } \xi_n = (-1)^n (e^{nb\pi/a} + 1 + 2 \sum_{k=1}^{n-1} e^{kb\pi/a}) , \quad n = 1, 2, \dots$$

Proof. By complete induction. By (9) and (10), the P-curve through the origin has the equations

$$x(t) = 1 - e^{-bt} \cos at , \quad y(t) = e^{-bt} \sin at$$

Putting $t = -\pi/a$ gives $\xi_1 = 1 + e^{b\pi/a}$, as the formula requires (the sum $\sum_{k=1}^{n-1}$ being of course vacuous in this case.) Now let it be assumed that the formula holds for $n = m$; it will be proved for $n = m + 1$. $(\xi_{m+1}, 0)$ is the end-point of an arc whose initial point is $(\xi_m, 0)$, and which is a P- or N-arc according as m is even or odd. Thus the equation of the arc is

$$x(t) = e^{-bt} (Ae^{iat} + Be^{-iat}) + (-1)^m$$

where

$$A = B = (-1)^{m+1} (\xi_m + 1) .$$

If one combine these equations with $t = -\pi/a$ and the assumed value of ξ_m , the result is

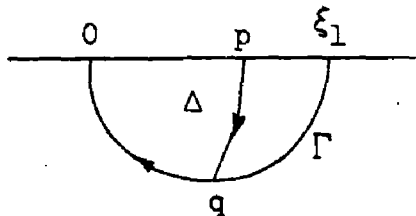
$$\xi_{m+1} = x(-\pi/a) = (-1)^m \left[e^{b\pi/a} (e^{mb\pi/a} + 2 \sum_{k=0}^{m-1} e^{kb\pi/a}) + 1 \right]$$

which was to be shown.

The letter Γ will be used to denote the P-arc joining the origin with $(\xi_1, 0)$; by the first lemma, the length of Γ is π/α .

Lemma 3. If $p \in R$, where R is the set which consists of the interval $0 < x < \xi_1$ on the x-axis and the interior of the region bounded by this interval and Γ , and if Δ is that path from p which is obtained by following the N-curve through p to Γ and then following Γ into the origin, $\tau(\Delta) < \pi/\alpha$.

Proof. It will certainly suffice to prove this for points on the axis only. To get what we want it will be convenient to find the coordinates of p and q by working out from the origin. The



equations of Γ in exponential form are

$$\begin{aligned} x(t) &= 1 - \frac{1}{2} e^{-bt} (e^{iat} + e^{-iat}) \\ y(t) &= \frac{i}{2} e^{-bt} (e^{-iat} - e^{iat}), \quad -\frac{\pi}{\alpha} \leq t \leq 0. \end{aligned} \quad (11)$$

The point q is given by these equations when t has some value $-\lambda$ ($0 < \lambda < \pi/\alpha$):

$$x_q = 1 - \frac{1}{2} e^{b\lambda} (e^{i\alpha\lambda} + e^{-i\alpha\lambda}), \quad y_q = \frac{i}{2} e^{b\lambda} (e^{i\alpha\lambda} - e^{-i\alpha\lambda}). \quad (12)$$

Regarding q as the initial point of the N-arc pq , the time necessary to reach p being $-\mu$ ($0 < \mu < \pi/\alpha$), one likewise obtains:

$$\begin{aligned} x_p &= e^{b\mu} (Ae^{-i\alpha\mu} + Be^{i\alpha\mu}) + 1, \\ y_p &= 0 = ie^{b\mu} (Ae^{-i\alpha\mu} - Be^{i\alpha\mu}), \end{aligned} \quad (13)$$

where $A = \frac{1}{2}(x_q + 1 - iy_q)$, $B = \bar{A}$.

The equation (13) implies that $Ae^{-i\alpha\mu}$ is real. Putting its imaginary part equal to zero and using (12) one gets

$$e^{b\lambda} \sin \alpha(\lambda + \mu) = 2 \sin \alpha\mu$$

Because $0 < \alpha\mu < \pi$, the right side, and therefore the left side of this equation is positive. This, together with the inequality $0 < \alpha(\lambda + \mu) < 2\pi$, implies $0 < \alpha(\lambda + \mu) < \pi$, which was to be shown.

With the aid of these three little lemmas it is possible to start finding minimal paths. As in the proof of Theorem 2, the procedure will be to examine one part of the lower half-plane after another until a unique minimal path from every point in this region has been found; the rest will follow by symmetry.

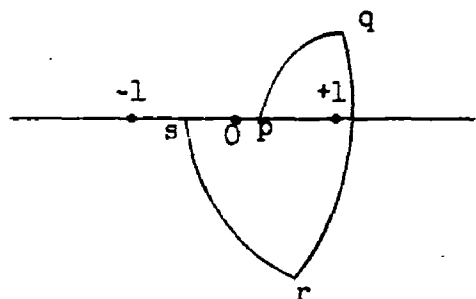
One more definition: A P-path is a path which begins with a P-arc, an N-path with an N-arc.

Theorem 3. If p is a point on the interval $0 < x < 1$ ($y = 0$), then given any P-path Δ from p which is of length $< \pi/\alpha$ one can find an N-path from p which is shorter than Δ .

Proof. Δ must be of the following type: it begins (by assumption) with a P-arc pq . This cannot return to the x-axis, for if it did it alone (and therefore the whole path) would have a length $\geq \pi/\alpha$, by Lemma 1; and this has been precluded. From q , Δ follows some N-arc qr which crosses the x-axis but, for the same reason as before, does not cross it again. (That it does cross the x-axis once may be seen as follows: it cannot stop short of the x-axis, or on the interval $0 < x < 1$, for this would give a corner of the wrong kind; and if it stopped on the axis to the right of the point $(1,0)$, the succeeding P-arc, by the corner condition, would necessarily return to the axis and therefore have a length $\geq \pi/\alpha$.) The N-arc qr is followed by a P-arc which must likewise cross the axis at some point s . The point s lies to the left of p , for otherwise Δ could not reach the origin from s without crossing itself. What Δ may be like beyond s does not matter.

If $\lambda = \tau(pq)$, $\mu = \tau(qr)$, $\sigma = \tau(rs)$, then in terms of these variables one can, by repeatedly applying (9) and (10), get the coordinates of the points q , r , s ; and from the resulting expressions one obtains, upon elimination,

$$x_s - 1 = e^{-\gamma\sigma} \left\{ [(x_p - 1)e^{-\gamma\lambda} + 2] e^{-\gamma\mu} - 2 \right\}, \quad \gamma = b - ai \quad (14)$$



Our object will be to show that, by holding p and s fixed and reducing λ to 0, we obtain a path which is shorter than Δ . For this purpose we shall prove

$$\frac{d}{d\lambda} (\lambda + \mu + \sigma) > 0$$

Since p and s are to be held fixed, λ can be taken as the sole independent variable in (14); and differentiating both sides of (14) with respect to λ leads to

$$\frac{d}{d\lambda} (\lambda + \mu + \sigma) = 2 \frac{e^{\gamma\sigma} \frac{d\sigma}{d\lambda} + 1}{(x_p - 1)e^{-\gamma\lambda} + 2} \quad (15)$$

(The denominator does not vanish; for when $b \neq 0$ the fact that $0 < \lambda < \pi/\alpha$ implies that it cannot even be real; and when $b = 0$ it is the sum of two positive terms.) This derivative is, of course, real; putting its imaginary part equal to zero gives

$$\frac{d\sigma}{d\lambda} e^{b\mu} [2 \sin \alpha\mu + (x_p - 1)e^{-b\lambda} \sin \alpha(\lambda + \mu)] = (1 - x_p)e^{-b\lambda} \sin \alpha\lambda. \quad (16)$$

One can use this to eliminate $\frac{d\sigma}{d\lambda}$ from the right member of (15) or, what is the same, from its real part. The result so obtained is

$$\frac{d}{d\lambda} (\lambda + \mu + \sigma) = \frac{2 \sin \alpha\mu [2 + (x_p - 1)e^{-b\lambda}]^2}{|2 + (x_p - 1)e^{-\gamma\lambda}|^2 [2 \sin \alpha\mu + (x_p - 1)e^{-b\lambda} \sin \alpha(\lambda + \mu)]}$$

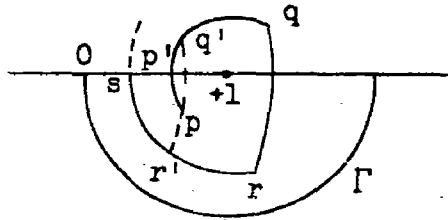
It follows from (16) that the second term in the above denominator is positive; thus the derivative has the sign of $\sin \alpha\mu$. When μ has its original value, this is positive, since $0 < \lambda + \mu + \sigma < \pi/\alpha$. So as λ is decreased the whole sum $\lambda + \mu + \sigma$ decreases, μ remains less than π/α , and the derivative remains positive. Thus we may shorten Δ by decreasing λ ; and λ may be decreased without changing the topology of the situation until one of two things happens:

I. r comes into coincidence with s . But this is impossible, for if it did occur the shortened path would then contain an N-arc $\bar{q}s$ which, since it intersects the x-axis twice, is of length $\geq \pi/\alpha$. This would contradict the assumption that $\tau(\Delta) < \pi/\alpha$.

II. q comes into coincidence with p ; that is, λ goes all the way to zero. This is just what was intended.

Corollary 3.1. If $p \in R$ (see Lemma 3), the conclusion of Theorem 3 remains true.

Proof. It is easy to see, by arguments like those used above, that a



sufficiently short P-path from p must be of the type $pp'qrs\dots$ shown in the adjoining sketch. By the proof of the theorem, one can show that decreasing $\lambda = \tau(p'q)$ shortens the whole path. This process may

be continued until one of two things happens:
I (described above) -- which is impossible for the same reason as before --
or:

II'. The arc qr comes to contain the point p . But in this case we get a path which is shorter than the original and consists of the closed loop $pp'q'p$ and an N-path from p . By simply eliminating the loop we get a still shorter path of the kind sought.

Corollary 3.2. If $p \in R$, any path Δ from p such that $\tau(\Delta) < n/\alpha$ which does not begin with an N-arc intersecting Γ may be replaced by a shorter one which does.

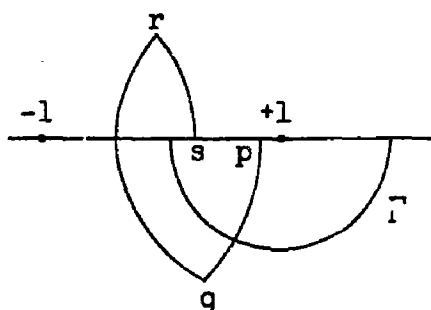
Proof. Let q be the initial point of the first P-arc of Δ . By assumption, $q \in R$. By applying Corollary 3.1 to q , we may shorten Δ by replacing $q\dots$ with a path from q which begins with a non-vacuous N-arc qq_1 . If $q_1 \notin R$, there is nothing more to do; if $q_1 \in R$, the procedure is to be repeated. Sooner or later the point q_n , which is the initial point of the first P-arc of the path after n such modifications, must lie on or below Γ ; for each point q_n is like r' (see the last sketch), and if all the points q_n lay in R , then all the corresponding points like s on Δ would lie to the right of the origin, which Δ could therefore never have reached. (It should be observed that infinitely many points q_n cannot occur, for if they did this would imply the existence of infinitely many pieces like $p'qrs$ in Δ , each outside the preceding one; and this, as it is not difficult to prove, would imply that the length of the whole path Δ was infinite.

Theorem 4. If $p \in R$, the unique minimal path from p is obtained by following the N-curve through p to Γ and then following Γ into

the origin. (If $p \in \Gamma$, the unique minimal path from p is obtained simply by following Γ into the origin.)

Proof. By Lemma 3, we know that there exists at least one path from p the length of which is less than π/α ; thus we need consider only paths satisfying this inequality. Then, in view of Corollary 3.2, we can further restrict our attention to those paths from $p \in R$ which follow the N-curve from p at least until it reaches Γ . We shall assume that p lies on the x-axis; for if the theorem is proved for this case, it will automatically follow for any point on the N-arc connecting p with Γ , and the totality of such points is (as p ranges over the interval) R .

Let Δ be such a path. It begins with an N-arc pq which crosses Γ but does not return to the axis, for this would make Δ too long. The corner q is followed by a P-arc qr which crosses the axis once (it cannot stop there -- unless $q = 0$ -- for this would force the succeeding N-arc to return to the axis, and this would again make Δ too long) but, for the same reason, does not return to it. The N-arc of Δ from r intersects the axis at some point s ; what Δ may do beyond s will be irrelevant.



The rest of the proof follows that of Theorem 3 almost to the letter. If one puts $\lambda = r(pq)$, $\mu = r(qr)$, and $\sigma = r(rs)$ and computes, using (9)-(10) repeatedly, the coordinates of q , r , and s in terms of λ , μ , σ , and the coordinates of p , one gets, after a little manipulation

$$x_s + 1 = e^{-\gamma\sigma} \left\{ [(x_p + 1)e^{-\gamma\lambda} - 2] e^{-\gamma\mu} + 2 \right\} \quad (\gamma = b - ci) \quad (17)$$

which corresponds to (14). Differentiating with respect to λ , one then gets

$$\frac{d}{d\lambda} (\lambda + \mu + \sigma) = 2 \frac{e^{\gamma\sigma} \frac{d\sigma}{d\lambda} + 1}{2 - (x_p + 1)e^{-\gamma\lambda}} \quad (18)$$

(As before, the denominator does not vanish.) Again separating real and imaginary parts, putting the latter equal to 0, and eliminating $\frac{d\sigma}{d\lambda}$ lead to:

$$\frac{d\sigma}{d\lambda} e^{b\mu} [2 \sin \alpha\mu - (x_p + 1)e^{-b\lambda} \sin \alpha(\lambda + \mu)] = (x_p + 1)e^{-b\lambda} \sin \alpha\lambda \quad (19)$$

$$\frac{d}{d\lambda}(\lambda + \mu + \sigma) = \frac{2 \sin \alpha\mu [(x_p + 1)e^{-b\lambda} - 2]^2}{|2 - (x_p + 1)e^{-\gamma\lambda}|^2 [2 \sin \alpha\mu - (x_p + 1)e^{-b\lambda} \sin \alpha(\lambda + \mu)]} \quad (20)$$

As in the earlier proof, (19) shows that the second factor of the denominator in (20) is positive; therefore (20) implies that the derivative has the same sign as $\sin \alpha\mu$. This quantity, however, is positive, for $0 < \mu < \lambda + \mu + \sigma < \pi/\alpha$. Therefore, as λ is decreased, so is $\lambda + \mu + \sigma$ (i.e., the path is shortened) and $\sin \alpha\mu$ remains positive. If λ is decreased until q lies on Γ , so that qr contains the origin, and if all the shortened path beyond 0 is cut off, a still shorter path from p is obtained, and it is exactly the path whose minimality was to be proved. This completes the proof.

Theorem 4 really constitutes the first step in an inductive argument, the whole of which will give the main result.

Theorem 5. If p is a point on the interval $|\xi_n| < x \leq |\xi_{n+1}|$, then the unique minimal path from p is that which consists of $n + 2$ arcs, the first of which is an N -arc of length λ ($0 \leq \lambda < \pi/\alpha$), the last of which is of length σ ($0 < \sigma \leq \pi/\alpha$), and the intervening ones of which are all of length π/α . (It will be shown that a path of this kind really exists and is unique.)

Proof. The proof will be by induction on n , $n = 0, 1, 2, \dots$. Theorem 4 gives the desired result for $n = 0$, where the interval is $(\xi_0, \xi_1) = (0, \xi_1)$. In this case, of course, the intervening arcs of length π/α do not occur.

The next step in the proof will be to determine the locus of the corners belonging to the paths described in the statement of the theorem. This will serve us in several ways. It is clear that the corners must be ob-

tainable in the following way: one starts at the origin and moves along Γ (or Γ^- , the reflection of Γ in the origin) for some interval of time $-\sigma$ ($0 < \sigma \leq \pi/\alpha$), then turns onto an N-(P-) arc and follows it for a time $-\pi/\alpha$, thence follows the succeeding P-(N-) arc for the same length of time, and continues to follow alternating P- and N-arcs until the n^{th} such has been traversed. The ends of these n^{th} arcs, as σ ranges over its interval, describe a certain curve E_n , which, I claim, is thus the locus of the first corners on the paths described in the theorem. It is only necessary to verify that this curve E_n is in the right place. To do this, parametric equations for E_n will be derived. If the n^{th} corner from the origin on the path described above is $(x_{nh}(\sigma), y_{nh}(\sigma))$, where $h = 0$ gives the corner on Γ or Γ^- , then

$$x_{nh}(\sigma) = (-1)^{n+h}(\rho^h \xi_h + |\xi_h|); \quad y_{nh}(\sigma) = (-1)^{n+h} \rho^h \eta_h \quad (21)$$

where $n = 0, 1, \dots$; $h = 0, 1, \dots, n$; $\rho = e^{b\pi/\alpha}$; $\xi_h = 1 - e^{b\sigma} \cos \alpha\sigma$; $\eta_h = -e^{b\sigma} \sin \alpha\sigma$. (For ξ_h see Lemma 2.)

One proves (21) by induction on h . The number n is really pertinent only in that it determines whether the path concerned begins on Γ or on Γ^- ; when n is even, it begins on Γ , and when odd on Γ^- . Thus

$$x_{n0}(\sigma) = (-1)^n(1 - e^{b\sigma} \cos \alpha\sigma), \quad y_{n0}(\sigma) = (-1)^{n+1} e^{b\sigma} \sin \alpha\sigma.$$

(Cf. (11), page 33.) In these equations we have (21) for $h = 0$. Now assume that (21) holds for $h = m$. $(x_{n,m+1}, y_{n,m+1})$ is the end of the arc of length $-\pi/\alpha$ from (x_{nm}, y_{nm}) and, as one can easily verify, is a P-arc if $n + m$ is odd, an N-arc if $n + m$ is even. Thus (9) and (10) give

$$x_{n,m+1}(\sigma) = -\rho(A + B) - (-1)^{n+m}, \quad y_{n,m+1}(\sigma) = i\rho(B - A)$$

where

$$A = \frac{1}{2} (x_{nm}(\sigma) + (-1)^{n+m} - iy_{nm}(\sigma)), \quad B = \bar{A}.$$

If one combines these equations with (21) for $h = m$, and uses Lemma 2, one obtains

$$x_{n,m+1}(\sigma) = (-1)^{n+m+1}(\rho^{m+1}\xi + |\xi_{m+1}|)$$

$$y_{n,m+1}(\sigma) = (-1)^{n+m+1}\rho^{m+1}\eta$$

as claimed.

In particular,

$$x_{nn}(\sigma) = \rho^n \xi + |\xi_n|$$

$$y_{nn}(\sigma) = \rho^n \eta$$

(22)

and these are the equations of E_n . Furthermore,

$$x_{nn}(0) = |\xi_n|, \quad x_{nn}(\pi/\alpha) = \rho^n(1 + \rho) + |\xi_{m+1}|$$

$$y_{nn}(0) = y_{nn}(\pi/\alpha) = 0$$

From this it is easy to see that E_n is a simple, semi-circle-like curve which lies below the x-axis and whose end-points are those of the interval $|\xi_n| < x \leq |\xi_{n+1}|$. E_n is, in fact, the curve obtained by magnifying Γ by the factor ρ^n and then translating the result to the right for a distance $|\xi_n|$. Now it is a simple matter to show the existence and uniqueness of the paths in question. For the existence, one needs only observe that the path obtained by going from p to E_n along an N-arc and then following the curve which, by the above construction, defined the point on E_n so attained, will suffice. On the other hand, any path of the type described must intersect E_n with an N-arc from p , and beyond this intersection behave like the path just described; but, for permissible values of λ , the N-curve from p intersects E_n only once; and this gives the uniqueness.

Now the proof proper can be begun. As before, one sets out by deciding how a path from p must behave. It begins, let us say, with an N -arc of length λ , where $\lambda \geq 0$. We shall momentarily assume:

$$\lambda \leq \pi/\alpha \quad (23)$$

Thus the first arc of the path ends at a point q on or below the x -axis, and from q the second arc qr , a P -arc emerges. The arc qr must cross the axis, but more will be assumed:

The arc qr intersects the x -axis in the interval $-|\xi_n| \leq x < -|\xi_{n-1}|$ (24)

The two assumptions (23) and (24) will be justified at the end of the proof.

Since the shortest path from p is being sought, we can also assume (and this requires no further justification) that once the path crosses this interval, say at the point v , it coincides with the shortest path from v , which is given by the inductive assumption and symmetry; for if it did not it could be replaced with one which did, and which would certainly be shorter than the given one. For such a path, therefore, r must be $(x_{n,n-1}(\sigma), y_{n,n-1}(\sigma))$ for some value of σ which depends on p and λ . In other words, by (21),

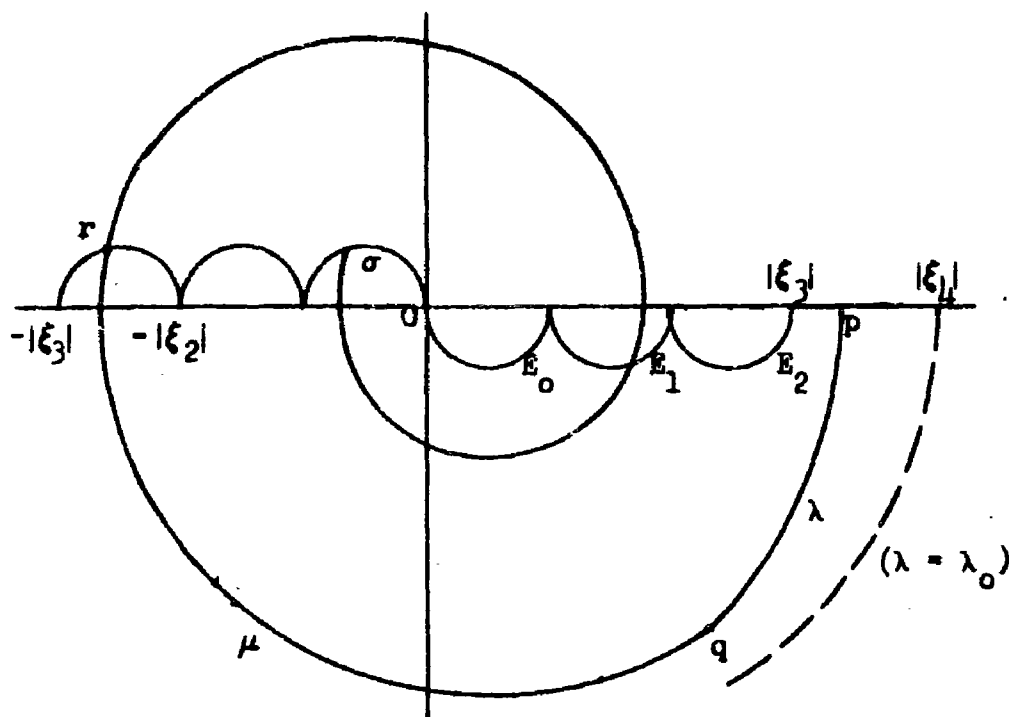
$$x_r = \rho^{n-1}(e^{b\sigma} \cos \alpha\sigma - 1) - |\xi_{n-1}|, \quad y_r = \rho^{n-1}e^{b\sigma} \sin \alpha\sigma$$

If the length of qr is μ , then using (9) and (10) for the P -arc of length $-\mu$ with the initial point r one gets:

$$x_q = e^{b\mu} (Ae^{-i\alpha\mu} + Be^{i\alpha\mu}) + 1, \quad y_q = ie^{b\mu} (Ae^{-i\alpha\mu} - Be^{i\alpha\mu})$$

where

$$A = \frac{1}{2}(x_r - 1 - iy_r), \quad E = \bar{A}.$$



Then following the N-curve from q for the time $-\lambda$, one gets

$$x_p = e^{b\lambda}(Ce^{-ia\lambda} + De^{ia\lambda}) - 1, \quad 0 = y_p = ie^{b\lambda}(Ce^{-ia\lambda} - De^{ia\lambda})$$

where

$$C = \frac{1}{2}(x_q + 1 - iy_q), \quad D = \bar{C}.$$

The result of combining all these equations and using Lemma 2 is

$$x_p + 1 = e^{\gamma\lambda} \left[(\rho^{n-1} e^{\gamma\sigma} - 2 \sum_0^{n-1} \rho^k) e^{\sigma\mu} + 2 \right] \quad (25)$$

where, as before, $\rho = e^{bn/a}$ and $\gamma = b - ai$. Our problem is to select the shortest path from those described, which satisfy (25). The length of any such path is $T = \lambda + \mu + \sigma + (n-1)n/a$. Thus the problem reduces to minimizing T with respect to (say) λ , x_p being held fixed. The range of variation of λ for a fixed x_p is $0 \leq \lambda \leq \lambda_0$, where λ_0 is that value of λ corresponding to $\sigma = \pi/a$. From this point on the argument will follow familiar lines. The result of differentiating both sides of (25) with respect to λ and suitably rearranging the terms is

$$\frac{dT}{d\lambda} = 2 \frac{\left(\sum_0^{n-1} \rho^k\right) \frac{d\sigma}{d\lambda} + e^{-\gamma\mu}}{2 \left(\sum_0^{n-1} \rho^k\right) - \rho^{n-1} e^{\gamma\sigma}} \quad (26)$$

The denominator does not vanish for those values of b and σ which have been admitted. Upon separating the real and imaginary parts of the right member of (26), and setting the latter equal to zero, one gets:

$$\left(\sum_0^{n-1} \rho^k\right) \frac{d\sigma}{d\lambda} = \frac{2(\sum \rho^k) e^{-b\mu} \sin \alpha\mu - \rho^{n-1} e^{b(\sigma-\mu)} \sin \alpha(\mu + \sigma)}{\rho^{n-1} e^{b\sigma} \sin \alpha\sigma} \quad (27)$$

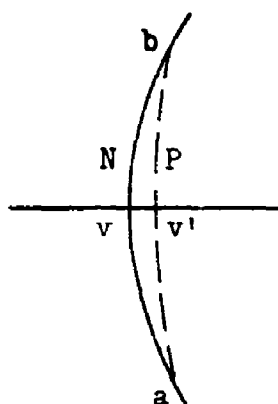
Thus the derivative (26) exists and is continuous for $0 < \sigma < \pi/\alpha$. Combining (26) and (27) yields the final relation

$$\frac{dT}{d\lambda} = \frac{2 \left[4 \left(\sum \rho^k\right)^2 - 4 \left(\sum \rho^k\right) \rho^{n-1} e^{b\sigma} \cos \alpha\sigma + \rho^{2(n-1)} e^{2b\sigma} \right] \sin \alpha\mu}{\rho^{n-1} e^{b(\sigma+\mu)} \sin \alpha\sigma \left| 2 \left(\sum \rho^k\right) - \rho^{n-1} e^{\gamma\sigma} \right|^2}$$

It is easy to see that all the factors involved, except for $\sin \alpha\mu$, are necessarily positive; therefore it is once again true that the derivative has the same sign as $\sin \alpha\mu$. We know that $\sin \alpha\mu = 0$ on E_n , for E_n is exactly the locus of q for $\mu = \pi/\alpha$. For λ smaller, i.e., for q above E_n , $\sin \alpha\mu$ and therefore $\frac{dT}{d\lambda}$ are negative; for λ larger, positive. Thus $\mu = \pi/\alpha$ gives T its minimum value, as was to be proved. (For $x_p = |\xi_{n+1}|$, $\lambda = 0$ and $\mu = \pi/\alpha$ give the only path from p of the type under consideration; thus it must be minimal, and the assertion is valid on the entire interval $|\xi_n| < x \leq |\xi_{n+1}|$.)

The proof will be complete as soon as the two assumptions (23) and (24) have been justified.

The assumption (23) will be justified by showing that any path Δ from p which violates (23) may be replaced by a shorter path which satisfies it. If (23) fails to hold for Δ , this means that the initial N -arc of Δ not only returns to the x -axis (on the negative half) but crosses it. Let v be the point at which this happens. The claim is that if one "shunts out" v by means of a short P -arc, a shorter canonical path is obtained. That such a shunting can be performed follows from the



fact that at a point like v the curvature of the P-curve is definitely less than that of the N-curve; this may be inferred from the given differential equations. If the P-arc introduced is short enough, it cannot cross Δ at any points but those indicated, so that it does not destroy the canonical nature of the path. That the path obtained in this way satisfies (23) is obvious; so it only remains to show that a shortening is truly effected.

This will follow from the result: If $p = (\xi, \eta)$, where $\xi < -1$, is a point near the x-axis, and if λ and λ' are the lengths of the shortest P- and N-arc respectively leading from p to the x-axis, then $\lambda < \lambda'$. Namely, if the shunting P-arc is sufficiently short, then both a and b (see the above picture) will satisfy the requirements on p ; thus the above proposition will imply that av' is shorter than av , and that $v'b$ is shorter than vb . Taken together, these show that $av'b$ is shorter than avb , which was to be proved.

To prove the result stated above, regard p as the initial point of the respective arcs. (Suppose, for the moment, that p lies below the x-axis.) By using (9) and (10), and the fact that the end points of the arcs are on the x-axis, one gets that the quantities

$$[(\xi - 1) - i\eta] e^{i\alpha\lambda} \quad \text{and} \quad [(\xi + 1) - i\eta] e^{i\alpha\lambda'}$$

are real. Setting their imaginary parts equal to zero gives

$$\tan \alpha\lambda = \frac{-\eta}{-\xi + 1} \qquad \tan \alpha\lambda' = \frac{-\eta}{-\xi - 1}$$

Since $\tan x$ is a monotone increasing function at points of continuity, and since λ and λ' are small,

$$\frac{-\eta}{-\xi + 1} < \frac{-\eta}{-\xi - 1} \quad \text{implies} \quad \lambda < \lambda'$$

as was to be shown. If p lies above the axis, the quantities η , λ ,

and λ' change sign, but these sign changes just cancel out to give the same conclusion.

The assumption (23) has thus been justified; in what follows (the justification of (24)) it will accordingly be assumed that all paths considered satisfy this assumption. In defense of (24), it will be shown that any path for which (24) fails must be longer than $\tilde{\Delta}_p$, the path which, according to the theorem, is minimal.

So let Δ be such a path. It begins with an N-arc of length λ ($0 < \lambda \leq \pi/\alpha$). If this arc ends at the point q , then q is the initial point of a P-arc which, by the argument adduced in support of (23) and symmetry, may be supposed to stop short of crossing the positive half of the x-axis. If r is the terminal point of this arc, there follows an N-arc starting at r which goes at least as far as the positive half of the x-axis (say at the point s). It may be that $r = s$. It follows from the definition of ξ_n that $|\xi_{n-1}| < x_s$. On the other hand, since Δ does not cross itself,

$$x_s < x_p \leq |\xi_{n+1}|$$

What happens to Δ beyond s is not certain; but sooner or later, in order to reach the origin, Δ must cross one of the two intervals

$$|\xi_{n-1}| < x \leq |\xi_n| \qquad -|\xi_n| \leq x < -|\xi_{n-1}|$$

(For by arguments of the type already given, Δ can cross neither the N-arc joining $(-|\xi_n|, 0)$ with $(|\xi_{n-1}|, 0)$, nor the P-arc joining $(|\xi_n|, 0)$ with $(-|\xi_{n-1}|, 0)$. Since these two arcs, together with the two intervals just described, form a simple closed curve surrounding the origin, the remark made must be true.) For momentary convenience, let $N(\Delta)$ be the number of times Δ crosses the x-axis after leaving p and before crossing one of these two intervals. (E.g., $N(\Delta) = 0$ would mean that Δ satisfies (24).) We next prove: if p is in the interval
 $|\xi_n| < x_p \leq |\xi_{n+1}|$, then

$$\frac{n\pi}{\alpha} < \tau(\tilde{\Delta}_p) \leq \frac{(n+1)\pi}{\alpha} \quad (28)$$

According to a formula on page 42

$$\tau(\tilde{\Delta}_p) = \lambda + \sigma + n\pi/\alpha$$

Thus it is only necessary to get bounds for $\lambda + \sigma$. Upon putting $\mu = \pi/\alpha$ in (25), one obtains:

$$x_p + 1 = e^{\gamma\lambda} \left[2 \left(\sum_0^n \rho^k \right) - \rho^n e^{\gamma\sigma} \right]$$

Since the imaginary part of the right member vanishes, we have

$$\rho^n e^{b\sigma} \sin \alpha(\lambda + \sigma) = 2 \left(\sum_0^n \rho^k \right) \sin \alpha\lambda$$

But on $\tilde{\Delta}_p$, $0 \leq \lambda < \pi/\alpha$ and $0 < \lambda + \sigma < 2\pi/\alpha$; these facts and the above equation imply $0 < \lambda + \sigma \leq \pi/\alpha$ which, in turn, implies (26).

We can say first that if $N(\Delta) \geq 3$, then Δ is longer than $\tilde{\Delta}_p$. For when $N(\Delta) \geq 3$, Δ begins with a curve $pqrs$ as described above, and s is followed by a curve $sq'r's'$ of the same sort. If w and v' are the points where the arcs qr and $q'r'$ intersect the x-axis, one can show* that each of the four pieces pqv , vrs , $sq'v'$, and $v'r's'$ has a length greater than $\pi/2\alpha$; and since the length of that part of Δ following s' is, by the inductive assumption and (28), greater than $(n-1)\pi/\alpha$, the length of Δ itself is greater than $(n+1)\pi/\alpha$ which, by (28), is in turn greater than $\tau(\tilde{\Delta}_p)$. Thus Δ is too long.

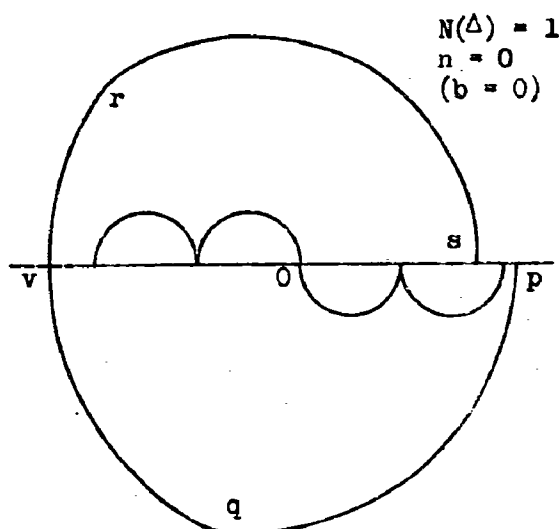
So the cases $N(\Delta) = 1$ and $N(\Delta) = 2$ are the only ones left.

* Consider pqv . If λ and μ are the lengths of pq and qv respectively, then working out the coordinates of v gives the equation

$$2 \cos \alpha\mu - (x_p + 1)e^{b\lambda} \cos \alpha(\lambda + \mu) = 1 - x_v$$

If it were true that $\lambda + \mu < \pi/2\alpha$, then the left member would be < 2 , since $\cos \alpha(\lambda + \mu) > 0$, while the right member would be > 2 , for $x_v < -|\xi_n| < -1$. This contradiction proves the assertion for pqv ; vrs , etc., are subject to the same argument.

When $N(\Delta) = 1$, $|\xi_{n-1}| < x_s \leq |\xi_n|$. This, together with the definition of ξ_n , can be made to imply:



$$-|\xi_{n+1}| \leq x_v < -|\xi_n|$$

In other words, v is a point like p , only on the opposite side of the origin. The part of Δ lying beyond v provides a path from v which (because of the assumed position of s) is of the type considered in the main part of this proof. It is therefore not shorter than $\tilde{\Delta}_v$, the path given by the statement of the theorem and symmetry as the minimal path from v .

It will thus suffice to prove that

the path Δ' obtained from Δ by replacing the part beyond v with $\tilde{\Delta}_v$ is longer than $\tilde{\Delta}_p$. A minor modification of the argument on pages 41, 42 43 will do it. Here we have exactly the same geometrical situation as there, except that it is as if the curve began from the interval

$|\xi_{n+1}| < x_p \leq |\xi_{n+2}|$ while in fact p is closer to the origin. These modifications do not vitiate the conclusion that $\frac{dT}{d\lambda}$ (T being the time length of Δ' as a function of $\lambda = \tau(pq)$) has the sign of $\sin \alpha\mu$.

Here, however, $\sin \alpha\mu > 0$, since $\mu < \pi/\alpha$. ($\mu = \pi/\alpha$ puts q on E_{n+1} ; so if $\mu \geq \pi/\alpha$, q would lie above this curve and force p to lie in $|\xi_{n+1}| < x \leq |\xi_{n+2}|$.) Thus $\frac{dT}{d\lambda} > 0$, and by deforming Δ' by decreasing λ (and keeping the path otherwise of the same type) one obtains a shorter path from p . The process may be continued until the path goes through $(-|\xi_n|, 0)$; but as soon as this happens the deformed path satisfies (24) and this, as we know, implies that it (and therefore Δ itself) is longer than $\tilde{\Delta}_p$. This was the desired conclusion.

If $N(\Delta) = 2$, $|\xi_n| < x_s < x_p$ and $-|\xi_n| \leq x_h < -|\xi_{n-1}|$, where h is the next intersection after s of Δ with the x -axis. The part of Δ after v , call it Δ_v , is either (1) of the type here considered*

* Satisfying (23) but not (24)

but with $N(\Delta) = 1$, or (2) of the type considered in the main part of the proof*, according as (1) - $|\xi_{n+1}| \leq x_v < -|\xi_n|$ or (2) - $|\xi_{n+2}| \leq x_v < -|\xi_{n+1}|$. (As before, because of the position of s , v can lie no farther to the left.) In either case, Δ_v is longer than $\tilde{\Delta}_v$, so that in place of Δ we may consider the path Δ' arising from Δ by replacing Δ_v with $\tilde{\Delta}_v$. If - $|\xi_{n+1}| \leq x_v < -|\xi_n|$, then Δ' , regarded as a path from p , satisfies $N(\Delta') = 1$, and therefore is longer than $\tilde{\Delta}_p$. If, however, - $|\xi_{n+2}| \leq x_v < -|\xi_{n+1}|$, then by (2d) the length of Δ' , which is greater than that of $\tilde{\Delta}_v$, is greater than $(n+1)\pi/\alpha$. But, again by (28), the length of $\tilde{\Delta}_p$ is less than this quantity. This is what was to be proved. In fact, this completes the whole proof of Theorem 5.

Corollary. If p lies above one of the curves E_n (see page 40) and below $y = 0$, the unique minimal path from p is that obtained by following the path described in Theorem 5 which passes through p .

Proof. (See the proof of B, page 27.)

The time has come to say something about the sign of the constant b , and the effect it has on the problem we have been discussing. The minimal path has been found for every point on $0 < x \leq |\xi_n|$, $n = 1, 2, \dots$. Now by Lemma 2, if $\rho = e^{b\pi/\alpha} \neq 1$,

$$|\xi_n| = \rho^n + 2 \sum_{k=1}^{n-1} \rho^k + 1 = (\rho^n - 1) \frac{\rho + 1}{\rho - 1}$$

Now when $b > 0$, $\rho > 1$ and therefore $|\xi_n| \rightarrow \infty$ as $n \rightarrow \infty$; i.e., the intervals $|\xi_n| < x \leq |\xi_{n+1}|$ cover the entire positive half of the x-axis. On the other hand, when $b < 0$, $\rho < 1$ and we have $|\xi_n| \rightarrow \frac{1+\rho}{1-\rho}$ as $n \rightarrow \infty$, so that in this case the set of intervals $|\xi_n| < x \leq |\xi_{n+1}|$ only covers $0 < x < \frac{1+\rho}{1-\rho}$. (When $b = 0$, $\rho = 1$ and $|\xi_n| = 1 + 2(n-1) + 1 = 2n \rightarrow \infty$.) From this it follows that:

When $b \geq 0$ there exists at least one path from each point in the plane.

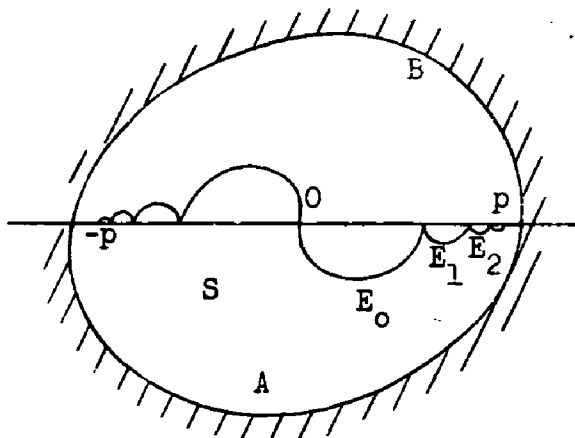
For the minimal paths from points on the positive half of the x-axis sweep out the whole plane; thus a path from any point may be obtained by taking that part of a minimal path through the point which lies beyond it.

* Satisfying both (23) and (24)

If $b < 0$ this is no longer true. If S denotes the interior of the set bounded by the P-arc A of length π/α joining $P = \left(\frac{1+\rho}{1-\rho}, 0\right)$ with $-P = \left(-\frac{1+\rho}{1-\rho}, 0\right)$ and the N-arc B joining the same two points (that these arcs exist may be verified by making a straight substitution in (9) and (10)), then:

When $b < 0$, there exists a path from p if and only if $p \in S$.

We first show that there exists at least one path from each point $p \in S$. Suppose that p lies below the curve C made up of the arcs E_n and the arcs arising from these by reflection in the origin. If the P-curve from p is drawn backwards ($t \rightarrow -\infty$), it must intersect one of the arcs E_n , for it surely crosses the x-axis to the right of the origin,



and without first crossing A , which is itself a P-arc. Let the first such point of intersection be q . Then if one follows the N-curve backwards (i.e., upwards) from q , one reaches a certain point r on the x-axis. Now the unique minimal path from r passes through p , and therefore that part of this path beyond p provides a path

from p . The corresponding device works if p is in the complementary part of S , by symmetry.

To show that no path can exist from a point outside S , we shall show that no path can cross the boundary of S going inwards. Consider B ; it is an N-arc, and therefore no path can cross it with an N-arc. But no path can cross it inwards with a P-arc, for all P-curves crossing B cross it moving outwards. Proof: By

$$\frac{dx}{dt} = -bx + ay \pm b \quad \frac{dy}{dt} = -ax - by \mp a, \quad (8)$$

the tangent vector at any point (x,y) on B has the components $(-bx + ay - b, -ax - by - a)$. Therefore the outward normal to B at this point has the components $(ax + by + a, -bx + ay - b)$. Also by (8), the tangent vector to the P-curve at the same point is $(-bx + ay + b, -ax - by + a)$.

The inner product of these last two vectors, which has the sign of the projection of the second on the first, is

$$(ax + by + a)(-bx + ay + b) + (-bx + ay - b)(-ax - by + a) = 2y .$$

Since $y \geq 0$ on B , this means that the projection is positive, i.e., that the P-curve crosses B moving outwards (except, of course, at the ends of B , where the two curves are tangent.) This proves the assertion made about B ; since A is subject to the symmetrical argument, the proof is complete.

Theorem 6. If p is a point in the fourth quadrant and below the curve made up of the pieces E_n ($n = 0, 1, \dots$), or in the third quadrant, and if a path from p exists, then the unique minimal path from p is that which follows the P-curve through p to the (negative half of the) x-axis and then proceeds according to the proposition arising from Theorem 5 and symmetry.

Proof. Consider any path Δ from p which might be minimal. It cannot cross any of the curves E_n and must therefore (having followed a possibly vacuous N-arc and then a possibly vacuous P-arc) cross the negative half of the axis, say first at the point v . By Theorem 5, we may suppose that Δ coincides with $\tilde{\Delta}_v$ beyond v . Now one should imagine Δ extended backwards to the axis by having added to it the N-arc preceding p which connects p with the axis, say at the point p' . The resulting path Δ' is of the type considered on pages 41-43, except for the possibilities that Δ' may cross itself on the arc pp' and that p' may not lie in the interval in which p was there supposed to lie. Both of these possibilities prove irrelevant. Moreover, μ (the length of the first P-arc qr of Δ' , which is also that of Δ) is less than π/α , by the position of q . Therefore, if T' is the length $\tau(\Delta')$ and λ is the length of $p'pq$ (the initial N-arc of Δ'), T' is a function of λ and the argument of pages 41-43 implies $\frac{dT'}{d\lambda} > 0$; thus by decreasing λ one decreases T' . (The derivative is actually discontinuous when v passes through one of the points $(-\xi_n, 0)$ during this contraction, but T' itself is a continuous function of λ .) If, in particular, one decreases λ until q comes into coincidence with p and then removes $p'p$

from Δ' , one obtains what must be the shortest path from p , and is in fact as claimed.

The main result for the spiral case can now be stated briefly and comprehensively. Let C denote the curve composed of the pieces:

$$x_n(\sigma) = \rho^n (2 - e^{b\sigma} \cos a\sigma) + \sum_{k=1}^{n-1} \rho^k + 1, \quad 0 < \sigma \leq \pi/a, \quad \rho = e^{b\pi/a}$$

$$y_n(\sigma) = -\rho^n e^{b\sigma} \sin a\sigma \quad n = 0, 1, 2, \dots$$

and of the pieces arising from these by reflection in the origin. C divides the set S from which paths can be drawn into an upper and a lower part. (S is as described on page 49 when $b < 0$, the whole plane in the contrary case.)

Theorem 7. If $p \in S$ and p is above (resp. below) C , the unique minimal path from p is obtained by following the N- (resp. P-) curve from p until it reaches C , then switching to the P- (resp. N-) curve through the point of intersection, then following this curve until it returns to C , then switching again, and so on, until the origin is attained. (In other terms, the unique minimal paths are obtained as solutions of (4) by taking $\phi(x, y) = -1$ above C and on that part of C to the left of the origin (within S) and $\phi(x, y) = 1$ in the rest of S .)

It should be recalled that all this is in terms of the oblique coordinates x and y . (See page 30.) The result is also valid for the original u and v if C is defined as follows: Let E_0 be the P-arc connecting the origin with the point $(\xi_1, 0)$, E_n be the curve obtained by magnifying E_0 by the factor ρ^n and translating the result $|\xi_n|$ units to the right ($n = 1, 2, \dots$), and E_n^- the curve obtained by reflecting E_n in the origin ($n = 0, 1, 2, \dots$); then $C = \left(\bigcup_0^\infty E_n \right) \cup \left(\bigcup_0^\infty E_n^- \right)$.

B. $|b| > 1$ (THE NODE CASE)

In this case the underlying equation is again

$$\frac{dx}{dt} = y$$

$$\frac{dy}{dt} = -1 - x - 2by$$

(6)

but now $|b| > 1$. (The marginal case $|b| = 1$, where the singular points are so-called degenerate nodes, is more closely related to this case than to the spiral case, but for technical reasons the proofs which follow do not happen to be directly extensible to it. However, the same general line of reasoning, appropriately recast, can be made to cover this case too. Because the amount of recasting necessary is small, and because the case is of no special significance, it will not be discussed in detail.)

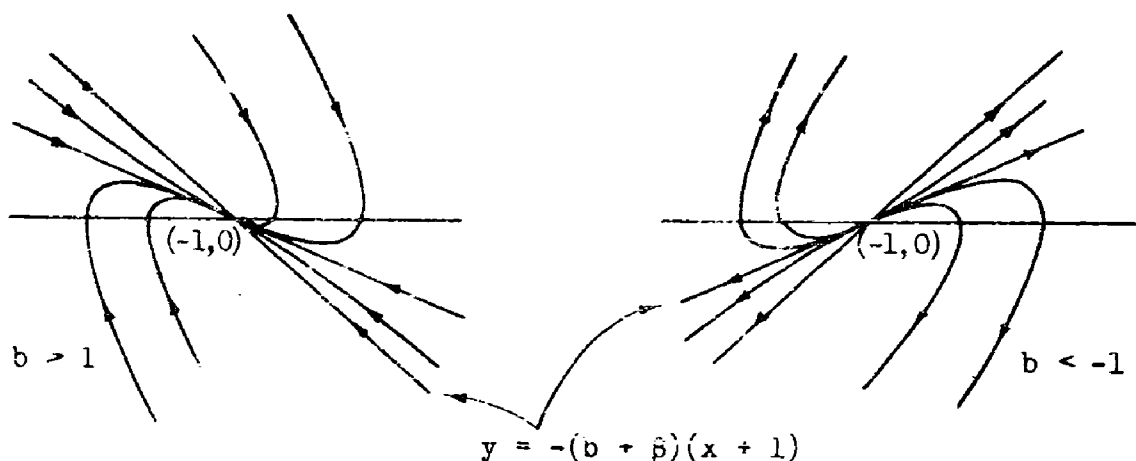
The P- and N-curves corresponding to (6) when $|b| > 1$ are given by the equations

$$\begin{aligned} x(t) &= e^{-bt} (Ae^{\beta t} + Be^{-\beta t}) + 1 \\ y(t) &= -e^{-bt} [(b - \beta)Ae^{\beta t} + (b + \beta)Be^{-\beta t}] \end{aligned} \quad (29)$$

where $\beta = \sqrt{b^2 - 1}$ and

$$A = \frac{1}{2\beta} [y(0) + (b + \beta)(x(0) + 1)] ; B = -\frac{1}{2\beta} [y(0) + (b - \beta)(x(0) + 1)] \quad (30)$$

The N-system consists of parabola-like curves which tend, when $b > 1$, to the point $(-1, 0)$ as $t \rightarrow \infty$ and, when $b < -1$, away from it; this kind of singular point is called a (stable or unstable) node. The special straight-line solution which, with the parameter eliminated, can be written $y = -(b + \beta)(x + 1)$ plays a special role; it will be called the major N-separatrix. Rough sketches of the N-system for the two signs of b are given below. To visualize the P-system, one should imagine all the curves moved two units to the right along $y = 0$.

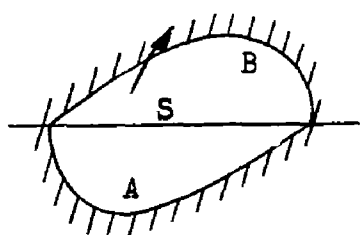


(The "degenerate node" occurs when $\beta = 0$; in this case the family of curves in the smaller sectors bounded by the linear curves collapses to a single straight line, and all the remaining curves are of the type which crosses the x-axis once.)

When $b \geq 1$ there are paths from each point in the plane; this will become clear when the minimal paths are described below. If $b \leq -1$, let A be the P-semicurve obtained by starting at $(-1,0)$ and letting $t \rightarrow -\infty$ and let B be the symmetrical N-semicurve. A and B together bound a certain open neighborhood S of the origin.

When $b \leq -1$, a path from p exists if and only if $p \in S$.

That a path from p can be found when $p \in S$ will be seen presently.



However, no path can enter S from the outside, for no path can cross either A or B moving inwards. Consider B ; it is itself an N-arc, so no path can cross it with an N-arc; it will thus suffice to prove that all P-curves crossing B do so moving outwards. (This will prove the

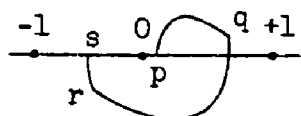
assertion for B , and the corresponding assertion for A will follow by symmetry.) By (6), $\frac{dy}{dx} = \frac{1-x-2by}{y}$. Therefore at any point on B ($y > 0$) the slope of the P-curve is greater than that of the N-curve, i.e., of B itself; since both curves move to the right, this means that each P-curve crosses B upwards, as was to be proved.

Lemma 4. If $1 \leq x_p < \infty$, no path from p can begin with a P-arc.

Proof. For $b \leq -1$, there exist no paths at all from such a point. When $b \geq 1$, the statement is obvious for $x_p = 1$, for then there is no P-arc from p . For $x_p > 1$, the P-curve through p goes downwards and to the left, as can be seen from (6); if a path from p began along this curve, it would be bound to stay on it until it crossed $y = 0$, by the condition on corners (page 24); but such a curve never does this -- it tends monotonely to the point $(1,0)$, remaining in the lower half-plane.

Lemma 5. Given any P-path (i.e., path beginning with a nonvacuous P-arc) from a point p on the interval $0 < x < 1$, one can find an N-path from p which is shorter.

Proof. Such a P-path must behave as follows: the P-arc with which it begins breaks off in a corner q above the axis, for the P-curve from p does not return to the axis in finite time. The following N-arc crosses the axis downwards (in fact, within the interval $0 < x < 1$) and, for the same reason, breaks off in a corner r below the axis. The succeeding P-arc extends back up to the axis, which it meets at a point s which, since the path cannot cross itself, must lie to the left of p ; however, s also



lies to the right of $(-1,0)$. The situation is thus as sketched. It will be shown, as previously, that when $\tau(pq)$ is decreased (p and s being held fixed), $\tau(pqrs)$ also decreases, so that in particular if q is brought down to coincide with p a path shorter than the given one will be obtained. This will be the N-path from p whose existence was claimed. (This proof differs from the corresponding one in the spiral case in that there is no need here to consider the possibility that r might come into coincidence with s before q reaches p ; for it is easy to see that r cannot come into coincidence with s at all.) If $\lambda = \tau(pq)$, $\mu = \tau(qr)$, and $\sigma = \tau(rs)$, it will thus suffice to show that

$$\frac{d}{d\lambda} (\lambda + \mu + \sigma) > 0$$

Using (29) and (30) repeatedly, one can compute the coordinates of q , r , and s in the usual way as functions of λ , μ , σ , and the coordinates of p ; from these one obtains

$$x_s = \left\{ [(x_p - 1)e^{h\lambda} + 2] e^{h\mu} - 2 \right\} e^{h\sigma} + 1 \quad (31)$$

where $h = -b + \beta$. Holding x_p and x_s fixed, differentiating with respect to λ , and rearranging the result, one obtains

$$\frac{d}{d\lambda} (\lambda + \mu + \sigma) = 2 \frac{e^{-h\mu} \frac{d\sigma}{d\lambda} + 1}{(x_p - 1)e^{h\lambda} + 2} \quad (32)$$

It is clear from the geometry of $pqrs$ that $\frac{d\sigma}{d\lambda} > 0$; therefore the numerator of the right member of (32) is positive. That the denominator is

positive when $h < 0$ ($b > 1$) follows from $x_p > 0$, $h\lambda < 0$. When $h > 0$ ($b > -1$), the denominator is also positive; were it not, it would follow from (31) that $x_s < -1$, and this is impossible.

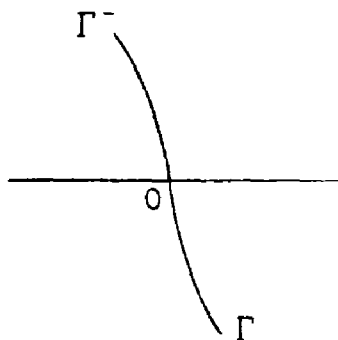
We again define Γ as that part of the P-curve through the origin which lies below the x-axis, and Γ^- as that part of the N-curve through the origin which lies above. Γ and Γ^- are symmetric to each other in the origin.

Lemma 6. If p is above Γ and below $y = 0$, any path from p which does not begin with an N-arc intersecting Γ can be replaced by a shorter path from p which does.

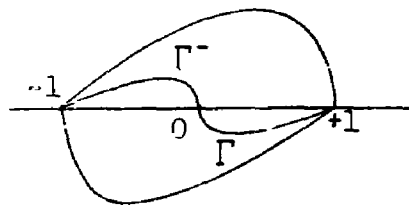
Proof. This Lemma follows from Lemma 5 in the same way that Corollary 3.2 followed from Theorem 3. (See pages 34-36.)

Theorem 8. In the case $g(x,y) = x + 2by$, $|b| > 1$, the unique minimal path from a point $p \in S$, S being the set from which paths exist, is determined in the following way: Let $C = \Gamma^- + 0 + \Gamma$; C is a simple curve which divides S into an upper and a lower part. If p is in the upper part, the minimal path from p is that obtained by following the N-curve from p until it intersects Γ , and then following Γ into the origin; if p is in the lower part, by following the P-curve from p until it intersects Γ^- and then following Γ^- into the origin. (In terms of $\phi(x,y)$, ϕ should be +1 below C and on Γ , -1 above C and on Γ^- .)

C is sketched for positive and negative b below. When $b > 1$, as already observed, S is the entire plane.



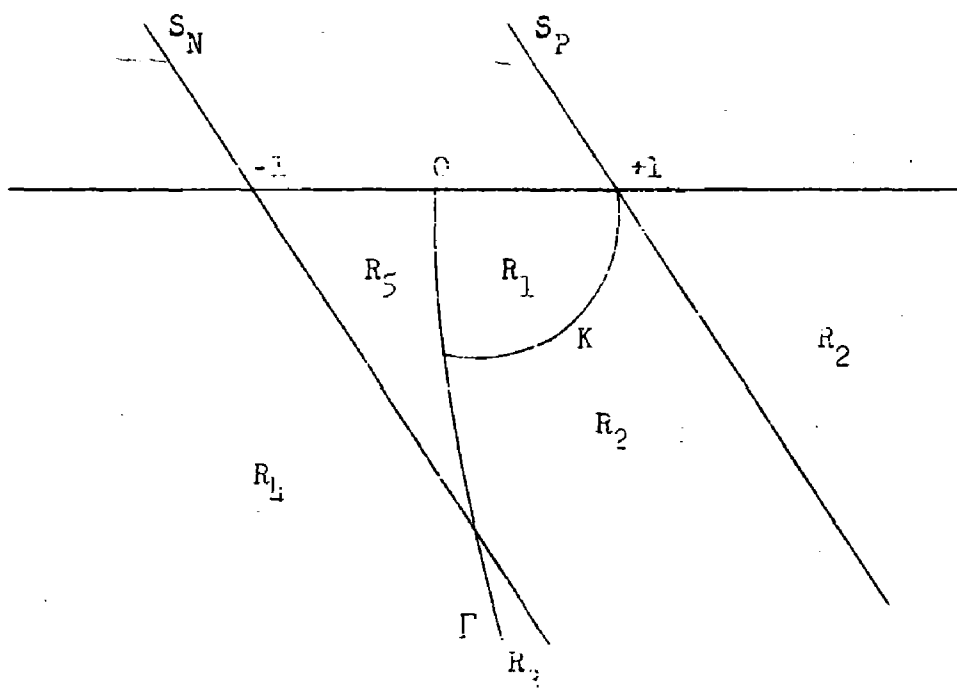
$b > 1$



$b < -1$

Proof. The theorem will be proved by considering various possible positions of the point p in the lower half-plane; the rest will follow by

symmetry. In the picture which follows (which is drawn for the case $b > 1$), S_N and S_P denote the major N- and P-separatrices respectively, and K denotes the N-arc connecting $(1,0)$ with Γ . When $b < -1$, the regions R_2 , R_3 , and R_4 do not occur, and much of the following proof (which has been written to fit the case $b > 1$) is irrelevant.



A. Suppose $p \in R_1$, this set being taken as closed. Let Δ be any path from p . By Lemma 6, it may be assumed that Δ begins with an N-arc pq which at least reaches Γ . q must lie below the axis, and the P-arc of Δ which begins at q must cross the axis (at or to the left of the origin); in fact, we may suppose, by the proposition which follows from Lemma 6 by symmetry, that this arc reaches Γ^- . Let it intersect Γ^- at the point r . The time lengths of the N-arcs pq and $r0$ will be denoted by λ and σ respectively; the time length of the P-arc pq will be denoted by μ . (N.B. It is not claimed that $r0$ is part of Δ ; this is merely an auxiliary arc introduced for convenience.) If, using (29)-(30), one computes the coordinates of q , r , and 0 (regarded as the end of the N-arc $r0$), one obtains

$$2 - Ce^{h\lambda} = e^{-h\mu} (2 - C'e^{h\sigma}) \quad , \quad (33)$$

where $h = -b + \beta$, $C = (b - \beta)\gamma_P + \kappa_P + 1$. The result of differentiating

both sides of (33) with respect to σ , which may be regarded as the independent variable giving $pqr0$, p being held fixed, is

$$\frac{d}{d\sigma}(\lambda + \mu) = \frac{2 \left(1 + \frac{d\lambda}{d\sigma}\right) + e^{h\mu} (Ce^{h\lambda} - 2)}{2 - Ce^{h\lambda}} \quad (34)$$

It will suffice to show that this quantity is positive; for this will imply that $\lambda + \mu$ has its minimum value when σ is as small as possible, i.e., when it vanishes; but this is exactly the case for the path claimed to be shortest, and for it $\lambda + \mu$ is the length of the entire path. Since, as usual, $\frac{d\lambda}{d\sigma} > 0$, it will in fact be enough to prove

$$0 < e^{h\mu} (2 - Ce^{h\lambda}) < 2, \quad (35)$$

from which the positivity of the right member of (34) follows at once. To prove (35), suppose first that $h < 0$ ($b > 1$). The first inequality of (35), since $h\lambda < 0$, would follow from $C \leq 2$. But this is true; for writing C out, transposing the 2 and multiplying both sides by the positive quantity $(b + \beta)$ gives the equivalent inequality

$$y_p + (b + \beta)(x_p - 1) \leq 0,$$

which is exactly the (true) statement that p lies to the left of or on the major P-separatrix. The other inequality of (35), since $h\mu < 0$, would follow from $C > 0$ which, in the same way, is equivalent with

$$y_p + (b + \beta)(x_p + 1) > 0,$$

but this is the statement that p lies to the right of the major N-separatrix, and this is also true. If $h > 0$ ($b < -1$), both of the inequalities in (35) are implied directly by (33).

B. Suppose now that $p \in R_2$, where this set is taken to include the adjoining sections of $y = 0$, S_N , and Γ , but to exclude K . The proof for this case follows simply from the preceding one. In A , all that was really used about p was that it lay between the two separatrices

and to the right of Γ . Thus the same proof applies verbatim to any point of R_2 to the left of the major P-separatrix. Suppose, then, that p lies to the right of this line. By Lemma 6, it is necessary to consider only those paths from p which begin with N-arcs intersecting Γ . Any such path must pass through the part of R_2 already dealt with (it being true that Γ and S_p do not intersect, for they are both P-curves); let p' , therefore, be any point on the N-curve from p which lies in that part of R_2 to the left of S_p ; by what has been shown, the shortest path from p' is of the type claimed; therefore the shortest path from p , consisting as it does of the N-arc pp' and the shortest path from p' , is again of the type claimed.

C. Suppose, finally, that $p \in R_3 + R_4 + R_5$, this set being taken open. By what has already been proved, we need to consider the following type of path only: it begins with an N-arc (which, if $p \in R_3$, must cross Γ but may otherwise be vacuous), say pq . From q the path follows a P-arc qr at least to the axis $y = 0$ and thence, by the results symmetrical to A and B , to Γ^- , which the path then follows into the origin. Thus we are now considering paths $pqr0$ as treated under A (where $pqr0$ was not in fact the true path considered), except that p is now in a different place. So (33) also holds here, but now $\lambda + \mu + \sigma$ is the actual length of the path being considered. By the usual argument, it will thus suffice to prove

$$\frac{d}{d\lambda} (\lambda + \mu + \sigma) > 0 ,$$

p being held fixed. In fact, the differentiation of both sides of (33) with respect to λ , followed by a rearrangement of terms, gives

$$\frac{d}{d\lambda} (\lambda + \mu + \sigma) = 2 \frac{e^{-h\mu} \frac{d\sigma}{d\lambda} + 1}{2 - Ce^{h\lambda}}$$

Again, it is evident that $\frac{d\sigma}{d\lambda} > 0$; thus the numerator is positive. That the denominator is positive follows (when $h < 0$) from the fact that p lies to the left of S_p or (when $h > 0$) from (33), as on page 57.

This completes the proof of Theorem 8.

VI. THE MINIMAL THEORY: $g(x,y) = -x + 2by$; SUMMARY

The only linear case yet unexamined is that corresponding to equation (2''') of page 3; it arises when $g(x,y) = -x + 2by$, and represents the physically improbable situation that there occurs not only the usual velocity damping, but also an "output damping" in the direction opposite to that which characterizes ordinary (simple or damped) harmonic motion.

The equations with which we have to deal are therefore

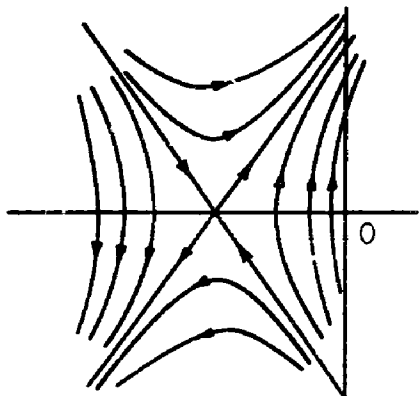
$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = -1 + x - 2by.$$

The P-system corresponding to these equations consists of hyperbola-like curves for which the point $(-1,0)$ is a saddle point, the separatrices being the lines

$$y = (-b \pm \delta)(x + 1), \quad \delta = \sqrt{b^2 + 1} > b.$$

Note that irrespective of the sign of b the two separatrices have slopes of opposite sign. In fact, the value of b plays no role at all in determining the qualitative behavior of the curves. This fact, together with the experience derived from the preceding sections, enables one to dispatch this case rather swiftly.

A sketch of the P-system is given below. The N-system is similar,



only moved two units to the right. The P- and N-curves are given explicitly by the equations

$$x(t) = e^{-bt}(Ae^{\delta t} + Be^{-\delta t}) - 1$$

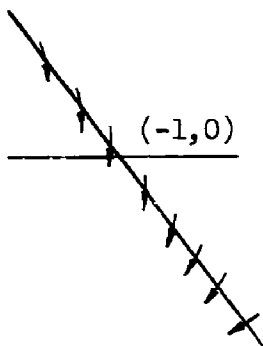
$$y(t) = -e^{-bt} [(b - \delta)Ae^{\delta t} + (b + \delta)Be^{-\delta t}]$$

where A and B are given by:

$$A = \frac{1}{2\delta} \left\{ y(0) + (b + \delta) [x(0) + 1] \right\}, \quad B = -\frac{1}{2\delta} \left\{ y(0) + (b - \delta) [x(0) + 1] \right\}.$$

We first note that a path from the point p exists if and only if p lies in the strip bounded by the parallel lines $y = -(b + \delta)(x \pm 1)$.

To get the necessity it will suffice to show that no path can cross either of the bounding lines into the strip. Consider the line $y = -(b + \delta)(x + 1)$; it consists of two P-curves, and therefore cannot be crossed by any P-curve whatsoever. Moreover, by the relation which subsists between the slopes of the P- and N-curves at a point, it can be seen that N-curves always cross the line going out of the strip. Thus no path can enter the strip along this line; the same holds for the other bounding line by symmetry.



That there do exist paths from any point within the strip will be seen below.

Now let Γ^- be, as before, that part of the N-curve through the origin which lies above the x-axis, Γ that part of the P-curve through the origin which lies below. $C = \Gamma^- + O + \Gamma$ is again a simple curve which divides the strip defined above into an upper and a lower part.

Theorem 9. When $g(x, y) = -x + 2by$, $b \geq 0$, the unique minimal path from any point p in the strip bounded by the two lines $y = -(b + \delta)(x \pm 1)$ is given as follows: if p lies above C (see the preceding paragraph), by following the N-curve from p to Γ and then following Γ into the origin; if p lies below C , symmetrically.

Proof. The proof is much like previous ones; so much so, in fact, that it is not worthwhile to give it in complete detail. The essential things to prove are:

(i) Given any P-path Δ from a point p on the interval $0 < x < 1$, one can find an N-path from p which is shorter than Δ .

From this follows, as in the derivation of Corollary 3.2, that for any point p below $y = 0$ and above Γ (and, of course, within the strip) it is only necessary to consider paths from p which begin with N-arcs intersecting Γ .

(ii) The unique minimal path from a point p on the interval $0 < x < 1$ is obtained by following the N-curve from p to Γ , and then following Γ into the origin.

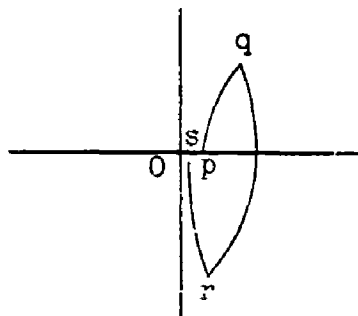
In view of (i), to prove this it will suffice to show that any path from p which intersects Γ with its initial arc, an N-arc, is longer than the claimed minimal path if it doesn't coincide with it.

Now let p be any point below the x-axis and within the strip, and let Δ be a path from it. Δ begins with an N-arc of length λ which, if p is to the right of Γ , must go as far as Γ , but otherwise may be vacuous. There follows a P-arc qr with q below the axis which must intersect the axis at exactly one point. This point of intersection, in fact, lies on the interval $-1 < x \leq 0$, so that the result arising from (ii) by symmetry implies that we may suppose that r lies on Γ^- , and that the rest of the path Δ is obtained by following Γ^- from r to the origin. Such a path is thus uniquely determined by λ (p being fixed), and may be written $\Delta_p(\lambda)$. To complete the proof for the lower half-plane (the rest will follow by symmetry) it will therefore suffice to prove:

(iii) The length of $\Delta_p(\lambda)$ is a strictly increasing function of λ .

Proof of (i). By the usual reasoning, one can see that Δ must

begin with an arc configuration like that shown. If $\lambda = \tau(pq)$, $\mu = \tau(qr)$, and $\sigma = \tau(rs)$, then we want to prove



$$\frac{d}{d\lambda} (\lambda + \mu + \sigma) > 0 .$$

Computing the coordinates of the corners, eliminating superfluous variables, and simplifying, one obtains

$$x_s = \left\{ \left[(x_p + 1)e^{h\lambda} - 2 \right] e^{h\mu} + 2 \right\} e^{h\sigma} - 1 .$$

where $h = \delta - b > 0$. If p and s are held fixed and both sides of this equation are differentiated with respect to λ , one gets

$$\frac{d}{d\lambda} (\lambda + \mu + \sigma) = 2 \frac{e^{-h\mu} \frac{d\sigma}{d\lambda} + 1}{2 - (x_p + 1)e^{h\lambda}}$$

Since $\frac{d\sigma}{d\lambda} > 0$, the numerator is positive. The denominator is also positive, for if it were not the above equation for x_s would make $x_s > 1$, and this is impossible.

Proof of (ii). Let Δ be any path from p ($0 < x_p < 1, y_p = 0$). As observed, we need consider Δ only if it begins with an N-arc pq which intersects Γ . The point q is followed by a P-arc qr which, by (i) and symmetry, may be supposed to cross or end on Γ^- . If the point at which qr intersects Γ^- is v , we shall consider the "virtual" path $pqv0$ instead of the true path $\Delta = pqr\dots 0$; for if it can be shown that $\tau(pqv)$ is a strictly increasing function of $\lambda = \tau(pq)$ (or, equivalently, of $\sigma = \tau(v0)$), (ii) will follow as in A, pages 56-57.

Again computing the coordinates of the corners q , v , and 0 (0 being regarded as the end of the arc $v0$), and so on, one gets

$$(x_p - 1)e^{h\lambda} + 2 = e^{-h\mu} (2 - e^{h\sigma}) \quad (33')$$

where $\lambda = \tau(pq)$, $\mu = \tau(qv)$, and $\sigma = \tau(v0)$. This is exactly equation (33), page 56, with $C = 1 - x_p$; therefore to get $\frac{d}{d\sigma} (\lambda + \mu) > 0$ which is what is needed, it will suffice to prove:

$$0 < 2 - Ce^{h\lambda} = (x_p - 1)e^{h\lambda} + 2 < 2.$$

The second inequality is obvious, since $x_p < 1$; the first follows from (33') and $h > 0$.

Proof of (iii). $\Delta_p(\lambda)$ is exactly like the "virtual" path $pqv0$ described above, except for the fact that p no longer lies on the x -axis. The equation corresponding to (33') is

$$2 - Ke^{h\lambda} = e^{-h\mu} (2 - e^{h\sigma}), \quad (33'')$$

where $K = hy_p + x_p - 1$. Differentiating (33'') gives

$$\frac{d}{d\lambda} \tau[\Delta_p(\lambda)] = \frac{d}{d\lambda} (\lambda + \mu + \sigma) = 2 \frac{e^{-h\mu} \frac{d\sigma}{d\lambda} + 1}{2 - Ke^{h\lambda}}$$

The numerator is again positive; that the denominator is positive follows from (33") and $h > 0$. This completes the whole proof.

SUMMARY OF THE MINIMAL THEORY FOR $g(x,y)$ LINEAR

It was pointed out on page 2 that the equation (2) can always be written in one of the three forms (2')-(2'''). In terms of the problem treated here, this means that when $g(x,y)$ is linear one may suppose that it has one of the three forms (i) $g(x,y) = by$; (ii) $g(x,y) = x + 2by$; (iii) $g(x,y) = -x + 2by$, where b is an arbitrary constant. The problem has now been completely solved for all three cases, and therefore for (2); the solution for (i) is Theorem 2 (page 26); for (ii), Theorems 7 (page 51) and 8 (page 55); for (iii), Theorem 9 (page 60).^{*} All these results may be summarized in the following form:

Theorem 10. When $g(x,y)$ is linear, the points from which paths exist for the corresponding system

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \phi(x,y) - g(x,y) \quad (4)$$

form a certain connected open set S containing the origin. There exists a unique simple curve C passing through the origin which divides S into an upper and a lower part, and such that if $\phi(x,y) = 1$ above C and on the part of C to the right of the origin, $\phi(x,y) = -1$ in the rest of S , then the solution curve of (4) from any point $p \in S$ is the unique minimal path from p . The set S and the curve C can be explicitly described when $g(x,y)$ is given.

Briefly, the problem stated at the bottom of page 3 has, when $g(x,y)$ is linear, a unique solution which can be explicitly described. This is the central result of this paper.

^{*} The physically important cases are (i) and (ii) with $b \geq 0$.

VII. $g(x,y)$ NONLINEAR

When it becomes necessary to attack a problem for which $g(x,y)$ is nonlinear, the most one should hope to be able to carry over from the last three sections is a few partial arguments and a point of view; for nearly everything that has been done for the linear case depended rather heavily on having explicit expressions for the P- and N-curves, and this is naturally out of the question for most nonlinear equations. However, some of the arguments which have been used did not really rely on those explicit expressions, and may be assembled in various ways to yield results under more liberal hypotheses. An example of such a result is given in Part A below.

In general, each nonlinear problem is likely to require a separate treatment, and usually a qualitative and partial discussion is the best one can legitimately expect. Such a discussion for the interesting case in which the F- and N-systems each contain a stable limit cycle of the relaxation oscillation type is given below in Part B.

A. $g(x,y)$ INDEPENDENT OF x

The results of IV (where $g(x,y) = by$) for $b \leq 0$ really depended only on the qualitative behavior of the solutions, and can therefore be generalized; one such generalization is:

Theorem 11. If (1) $g(x,y)$ has the following properties:

(i) $g(x,y)$ is independent of x ; i.e., $g(x,y) = f(y)$;

(ii) $f(-y) = -f(y)$;

(iii) $f(-K) = 1$, $f(y) < 1$ for $|y| < K$, K being some positive constant;

and if (2) R is the open strip bounded by the two lines $y = \pm K$; then there exists a unique minimal path from any point $p \in R$, determined as in Theorem 2.

Examples. Some functions satisfying (i)-(iii) are: $g(x,y) = Cy^n$, $C < 0$, $n = 1, 3, 5, \dots$, $K = -C^{-1/3}$; $g(x,y) = -\sin(\alpha y)$ ($K = \pi/2\alpha$) .

Proof. The assumption (i) implies that any P-curve can be obtained

from any other by translating it along the x-axis; (ii) gives the condition for symmetry, as discussed at the end of III. The crucial assumption is (iii). The P-curve through the origin (which will be taken as a typical P-curve) satisfies the equations $\frac{dx}{dt} = y$, $\frac{dy}{dt} = 1 - f(y)$. At the origin the motion is upwards and, as it leaves the origin, tending to the right. By (ii) and (iii), $f(K) = -1$; therefore $f(y) < 1$, and the function $1 - f(y)$ attains a positive minimum ϵ , on $0 \leq y \leq K$. From this, by the second differential equation,

$$y = \int_{t_0}^t [1 - f(y)] dt \geq \epsilon (t - t_0) \quad (0 \leq y \leq K)$$

where t_0 is the value of the parameter representing the origin. But this means that for some value of t (e.g., $t = t_0 + K/\epsilon$) the curve will have crossed the line $y = K$. Thus the P-curve through the origin moves monotonely upwards and to the right from the origin until, after some finite interval, it intersects $y = K$.

By applying a similar argument to any interval $-K + \delta \leq y \leq 0$ ($0 < \delta < K$), we see that the curve must cross each line $y = -K + \delta$ as $t \rightarrow -\infty$ and must therefore come arbitrarily close to the line $y = -K$. But it cannot cross this line, which is itself a P-curve. Once the curve has crossed the line $y = -K + \delta$ (for decreasing t) it must remain within the strip $-K < y < -K + \delta$, for $\frac{dy}{dt} = 1 - f(y) > 0$ in this strip. Moreover, since $\frac{dx}{dt} = y < -K + \delta < 0$ in this strip, the curve goes infinitely far to the right as $t \rightarrow -\infty$.

Thus it stands verified that the P-curve through the origin, and therefore all P- and N-curves, have exactly those properties of P- and N-curves used in proving Theorem 2 for $b < 0$. The proof of that theorem, accordingly, can be applied directly here, and gives the result stated.

Corollary 11.1. The conclusion of Theorem 11 remains true if one replaces (iii) and (2) by:

(iii') $f(y) < 1$, for $-\infty < y < \infty$;

(2') R is the entire plane.

Proof. Everything is as before, except that now the P-curve through the origin goes strongly to infinity in the sense that if $x(t)$ and $y(t)$ represent this curve, $x(t) \rightarrow \infty$ and $y(t) \rightarrow -\infty$ as $t \rightarrow -\infty$, while $x(t) \rightarrow \infty$ and $y(t) \rightarrow \infty$ as $t \rightarrow \infty$. The curve thus behaves qualitatively like a parabola with its vertex at the origin and $y = 0$ as its axis. The proof of Theorem 2 for $b = 0$ consequently works here.

Corollary 11.2. The conclusion of Theorem 11 remains true if one replaces (iii) by:

(iii") $f(-K) = 1$ and $f(y) < 1$ for $|y| < -K$, K being some negative constant.

Proof. In the same way that Theorem 11 corresponds to Theorem 2 with $b < 0$, this corresponds to Theorem 2 with $b > 0$. There is the difference that Theorem 11 accounts for all the points from which paths exist, while for a function satisfying (iii") there may very well exist points outside the strip $|y| < -K$ from which paths do exist but for which no minimal paths are described. (Indeed, this happens when $f(y) = by$, $b > 0$, as we have seen.) The reason for this incompleteness is that it was at just this stage in the proof of Theorem 2 that the explicit formulae for the P- and N-curves were used.

B. LIMIT CYCLES

When $g(x,y)$ is nonlinear, limit cycles can occur among the P- and N-curves. As is well known, the close study of these limit cycles -- especially as regards their exact quantitative characteristics -- presents great difficulties. The object of this discussion will be to show one way limit cycles can occur in a manner relevant to our problem, and then to say something about the corresponding choice of $\phi(x,y)$.

We shall consider the generalized van der Pol equation

$$\frac{d^2x}{d\tau^2} + \mu f(x) \frac{dx}{d\tau} + x = 0 \quad (36)$$

where $f(x) \in C^1$ and μ is a real, positive parameter. If $G(x)$ is defined by

$$G(x) = - \int_0^x f(u) du ,$$

then (upon putting $t = \mu \tau$) the equation (36) can be written

$$\frac{dx}{dt} = G(x) - y \quad \frac{dy}{dt} = \frac{1}{\mu^2} x .$$

In this situation a theorem due to LaSalle (LaSalle (1)) states that:

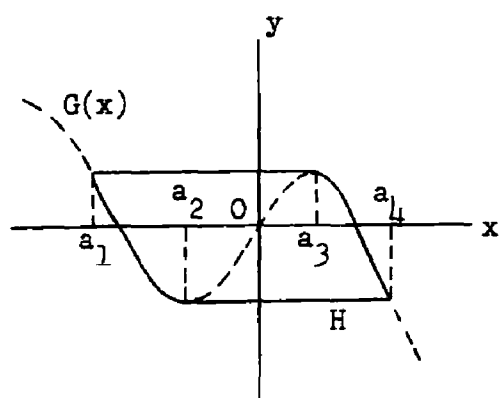
If there exist four numbers $a_1 < a_2 < 0 < a_3 < a_4$ such that

(a) $G(a_1) = G(a_3)$ and $G(a_2) = G(a_4)$,

(b) $G(a_2) \leq G(x) \leq G(a_3)$ for $a_1 \leq x \leq a_4$, and

(c) $G'(a_1) < 0$ and $G'(a_4) < 0$,

then for $\mu > \mu_0 > 0$ there exists a unique stable limit cycle in a certain neighborhood of the curve H (see the figure); as $\mu \rightarrow \infty$, the limit cycle converges to H .



What we are really concerned with is not (36), but rather the corresponding equations

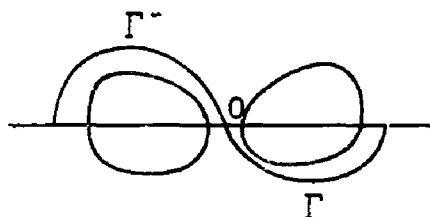
$$\frac{d^2x}{d\tau^2} + \mu f(x) \frac{dx}{d\tau} + x = \pm 1 \quad (37)$$

Because of the nonlinearity, the existence of a periodic solution of (36) does not guarantee the existence of a periodic solution for either of the equa-

tions (37). What one would like to see happen, however, is that there occur a P-limit cycle lying slightly to the right of the origin and an N-limit cycle slightly to the left, so that each could act as a "big focus" and the curves Γ and Γ^- , defined as in the spiral case (see page 33), would be spiral arcs acting as they did there.* One would then

* One could also seek P- and N-cycles each just barely containing the origin at its left or right extreme respectively; this would lead to considerations similar to those which follow.

suppose that the minimal paths from points in some neighborhood of the



origin would be obtained by taking $\phi(x,y) = +1$ above C and on Γ , -1 below C and on Γ^- within the neighborhood. Indeed, the very purpose here is to imitate the spiral case, having limit cycles in place of foci.

$$C = \Gamma^- + 0 + \Gamma$$

If we put $x = u \pm 1$, (37) becomes

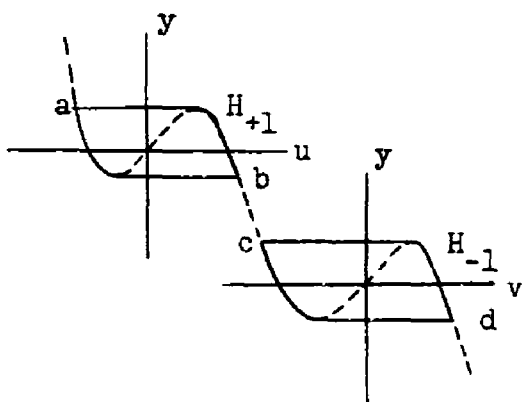
$$\frac{d^2 u}{d\tau^2} + \mu f(u \pm 1) \frac{du}{d\tau} + u = 0, \quad (36^\pm)$$

and this is subject to LaSalle's theorem. Putting

$$G_{\pm 1}(u) = - \int_0^u f(v \pm 1) dv,$$

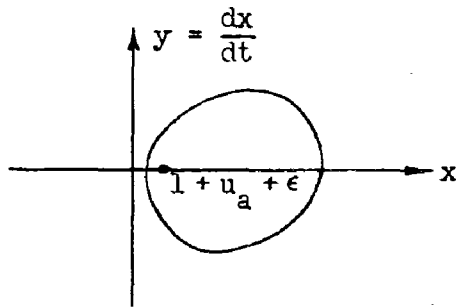
one gets the functions $G_{+1}(u)$ and $G_{-1}(u)$ corresponding to $G(x)$ for the P- and N-systems respectively. It is easy to see that the curve of $G_{-1}(u)$ can be obtained by moving that of $G_{+1}(u)$ two units to the left along $y = 0$ and then raising or lowering it until it again passes through the origin.

It follows from LaSalle's theorem that both the equations (36 $^\pm$) will give limit cycles in the u,y -plane (y being as above, and not identified with $\frac{du}{dt}$ as previously) as described when μ is sufficiently large if each of the functions $G_{\pm 1}(u)$ satisfies a set of conditions (a)-(c), i.e., if their common curve has the general shape of that in the following figure.



If u_a (the abscissa of the point a with respect to the u -axis) satisfies $u_a > -1$, then for a given $\epsilon > 0$ and μ sufficiently large the limit cycle corresponding to H_{+1} will have its left extreme point p on the interval $-1 < u_p < u_a + \epsilon$. Similarly, the limit cycle corresponding to H_{-1} will have its right extreme point q on the interval $v_d - \epsilon < v_q < 1$, if $v_d < 1$.

The function $u(t)$ which represents the limit cycle corresponding to H_{+1} is thus a periodic function whose minimum value lies in the interval $(-1, u_a + \epsilon)$. The corresponding P-curve (in the original $x, \frac{dx}{dt}$ -plane) for (37) is therefore a limit cycle whose left extremity lies on the interval $(0, 1 + u_a + \epsilon)$; this point must lie on the x-axis because $\frac{dx}{dt} = y \neq 0$ off



the axis. This limit cycle is consequently of the kind illustrated, as was desired. An N-limit cycle of the corresponding type on the other side of the origin is obtained in the same way.

It is not certain that these limit cycles are the only ones in the P- and N-systems, or even that they are the only ones passing through a prescribed neighborhood of the origin; but when they are, the P- and N-curves behave near the origin in the manner described on the preceding page.

In order for all this to happen, the function $G_{+1}(u)$ must have four relative extrema, and therefore $f(x)$ must have at least four zeros and be, if a polynomial, of degree four or higher. (A simple transcendental function which has the necessary properties is

$$f(x) = \pi \cos \pi x + cx \quad ,$$

where c is a constant which must be properly chosen; it gives

$$G_{+1}(u) = \sin \pi u - cu \quad ,$$

a function whose graph is of the type shown.)

It was once thought that the introduction of such limit cycles might make for better response of the system in the vicinity of the origin than linear systems could provide; that the presence of the limit cycles might accelerate the motion along the P- and N-curves nearby. But this violates our basic principle (which, to be sure, has not been rigorously demonstrated) that the best control should be that which uses the full magnitude of the available control force at all times; for if a certain amount of force is

available to be put into the first order term in (37), it would then be better to take it at its full strength and combine it with the control force already present (in the form $+1$) on the right-hand side, than to use it partially and continuously in the form of the function $\mu f(x) \frac{dx}{d\tau}$.

However, if an equation like (37) is dictated by circumstances, and the function $f(x)$ cannot be altered at will, a discussion of the above sort must be undertaken. In the case which has been considered the limit cycles occur in what, from the standpoint of tractability, is probably the simplest way; but even so it is complex enough. But even when such a discussion can be carried out in sufficient detail, it is still only prefatory to the treatment of the problem actually at hand, that of finding the minimal paths. Except for conjectures based on analogies with the linear case, nothing whatsoever has been discovered along these lines.

APPENDIX I. VELOCITY CONTROL*

In section II a description was given of the conclusions reached in previous investigations of the behavior of the solutions of the equation

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + x = \phi\left(x, \frac{dx}{dt}\right) \quad (0 < b < 1) \quad (2')$$

when $\phi(x, y)$ is of the special type $\phi(x, y) = \text{sgn}(Kx + My)$, K and M being constants. The same group of investigators has also studied the similar problem in which the control force has a derivative like this; i.e., the problem associated with the equation

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + x = \psi \quad (0 < b < 1) \quad (38)$$

where ψ is a continuous function of t and, at points where $Kx + My + N\psi \neq 0$, satisfies

$$\frac{d\psi}{dt} = \text{sgn}(Kx + My + N\psi) \quad , \quad y = \frac{dx}{dt} \quad (39)$$

K , M , and N being constants.

In the former problem, ϕ along any solution was a step function alternating between the values ± 1 ; here ψ along any solution is a continuous "sawtooth" function whose graph consists of linear pieces of slope ± 1 . Such a system (called velocity control) has the practical advantage that, without being essentially more difficult to design, it gives a smoother operation. This, in fact, is its principal virtue; it does not claim to give, for a bounded control force, a rapidity of response comparable to that of the discontinuous, "position control" arrangement which has been discussed. Still, one might consider the problem of replacing the function $\text{sgn}\left(Kx + M\frac{dx}{dt} + N\psi\right)$ by a more general function of $\left(x, \frac{dx}{dt}, \psi\right)$ which takes on only the values ± 1 in such a way as to obtain minimal paths in the $\left(x, \frac{dx}{dt}, \psi\right)$ -space and

* The appendices deal with questions which lie outside the domain proper to the main text, but which nevertheless are of a kindred nature and should be discussed for the sake of completeness.

thus the most rapid response within the limits of the system. This problem has not been touched. What has been done (the most recent account is in Flügge-Lotz (2), Chapter 5) is to make a graphical study of the solutions of (38)-(39) after the manner of that described in section II above. The several cases that occur corresponding to the different possible combinations of signs on the constants K , M , and N are given separate treatments.

The technique used to implement the investigation is first to introduce the auxiliary variable

$$y(t) = x(t) - \psi(t)$$

and to consider solutions in the $\left(y, \frac{dy}{dt}, \psi\right)$ -space, where they consist of arcs whose projections in the $\left(y, \frac{dy}{dt}\right)$ -plane are arcs of logarithmic spirals. Then the solutions are considered in terms of their projections in this and a certain other plane, whereby all the desired information can be obtained. One again encounters end points, start points, periodic solutions, etc. It may also happen that a solution has a last corner beyond which it moves off to infinity; the purpose of introducing the term in ψ in the argument of the signum function is to avoid this phenomenon. Further details will not be given here.

APPENDIX II. MULTIPLE MODE CONTROL

Those interested in the practical applications of the theory for the case in which $g(x,y)$ is linear will have noted that the subject of time lag was ignored in III-VI. The present appendix will have something to say about this and related matters.

It is easy for one to convince himself that the systems of minimal paths discovered in those sections (and summarized in Theorem 10, page 63) are "structurally stable with respect to time lags" in the sense that the presence of a sufficiently small time lag does not affect the essential over-all behavior of the solution curves. This follows from the fact that the time length of the minimal path from a point depends continuously on the position of the point, from the character of C with relation to the P- and N-curves, and so on. In fact, the only place where a small time lag can seriously mar the qualitative situation is at the origin. There a time lag has the effect of causing a solution to overshoot the origin slightly instead of ending there, have a corner, overshoot in the reverse direction, have another corner, and repeat the act indefinitely. (Cf. page 16.)

Perhaps the first means of avoiding this which suggests itself is that of slightly altering the curve C so that the time lag is anticipated and pre-corrected; but this is not really feasible, for the time lag is a complex and variable thing which cannot be predicted precisely or be expected to be always the same. Indeed, the assumption that a time lag is simply a clean, sharp delay in the occurrence of a corner is itself a considerable distortion of the real state of affairs.

Far better is the plan of disconnecting the control entirely as soon as the solution has entered a satisfactorily small neighborhood of the origin, and replacing it there either by a control of a different sort, or by no control at all, letting the system run free.

If, for instance, the system in question is given by the equation

$$\frac{d^2x}{dt^2} + 2b \frac{dx}{dt} + x = +1, \quad (0 < b < 1)$$

when the control force $+1$ is removed the solution curve spirals into the

origin. (The very function of the control force \ddot{x} is to accelerate this process.) If, therefore, the mechanical system is so designed that the force represented by \ddot{x} is switched in accordance with Theorem 10 (page 63), except that in some small neighborhood of the origin in the x, y -plane ($y = \frac{dx}{dt}$) it is removed entirely, then a typical solution curve would behave in the following way: it would move rapidly in from its initial point and then, having passed through a finite number of corners (in any but the spiral case, at most one) it would enter the neighborhood mentioned; once this had happened the solution would spiral gently down to the origin. In the presence of a time lag the solution curve would not behave differently. This arrangement has the valuable properties that:

- 1) it reduces any error, together with its first derivative, almost to zero very rapidly, and thus accomplishes the purpose of the system within arbitrarily small tolerance limits:
- 2) it prevents high-frequency oscillations back and forth around the origin such as would occur in the presence of a time lag otherwise;
- 3) it prevents the system from being hypersensitive to trifling disturbances from the zero-error state.

Such an arrangement gives what is called dual mode control; this is what arises in the conventional theory of servomechanisms when the feedback circuit has a threshold of sensitivity below which it gives no output at all. (See for example the Appendix in MacColl (1).) In the context of our minimal theory, its principal interest lies in assuring the mathematician that conclusions reached disregarding time lags need not be utterly useless when confronted with practical demands. But the problem of choosing the "right" dual mode arrangement in a particular situation is not a mathematical one.

In general, a multiple mode control might be defined as a control system in which different kinds of control are to be applied in different parts of the phase plane. Such arrangements are frequently desirable when one or more of the assumptions underlying the problem stated in I fail. An instance of this is discussed at some length in the next appendix.

One should perhaps include with multiple mode controls those subject to special restraints of various kinds. For instance, it may happen that physical limitations prevent the magnitude of the derivative of the controlled variable (in terms of the example of page 1, the angular speed of the motor) from exceeding a certain value, K , so that any solution must be contained in the horizontal strip $|y| \leq K$ in the phase plane. In this case there is a multiple mode control in the sense that the control is not meaningfully described at all outside the strip; if a path reaches one of the boundaries of the strip (necessarily from the interior), it must follow this boundary for some distance in the proper direction and then return to the interior and, eventually, the origin. It is easy to see that for the case in which $g(x,y) = \dot{y}$ by the curve C should be the same (within the strip $|y| \leq K$) as if y were unconstrained; what happens in more complicated cases, for example the spiral case, is not clear.

APPENDIX III. COULOMB DAMPING*

The whole problem with which we have dealt derived much of its importance from the assumption that the best behavior (in terms of response time) should be obtained from a servomechanism when the full strength of the power source is used; but this "full strength" may vary for different states of the system, instead of remaining constant as we have supposed. In this appendix such a case will be discussed.

In terms of the example of page 1, it may be possible to apply to the motor a second control force in the form of "Coulomb damping," which is characterized by being constant in magnitude but opposite in sign to the derivative of the output; the equation for the system could therefore be written

$$I \frac{d^2 x}{dt^2} + R \frac{dx}{dt} = K \cdot f \left(x, \frac{dx}{dt} \right) - C \cdot h \left(x, \frac{dx}{dt} \right) \cdot \operatorname{sgn} \left(\frac{dx}{dt} \right), \quad (40)$$

where everything is as in equation (1), page 1, except for C , which is some positive constant, and $h \left(x, \frac{dx}{dt} \right)$, which takes on only the values ± 1 and 0 and represents the instruction from the control as to whether or not the Coulomb damping is to be applied at any particular time. The right hand member of the above equation can be written simply $F \left(x, \frac{dx}{dt} \right)$, where this function assumes only the values

$$\begin{aligned} K, \quad -K, \quad K - C, \quad -K - C & \quad \left(\text{for } \frac{dx}{dt} > 0 \right) \\ K, \quad -K, \quad K + C, \quad -K + C & \quad \left(\text{for } \frac{dx}{dt} < 0 \right) \\ K, \quad -K & \quad \left(\text{for } \frac{dx}{dt} = 0 \right) \end{aligned}$$

Now the natural question to ask is: How should F be chosen so that all solutions of (40) will go to the origin in the $x, y, \left(= \frac{dx}{dt} \right)$ -plane in minimum time? If the assumption mentioned above is valid, it should again be true that best results can be obtained by using only the extreme values of F available at any point. This is equivalent to saying that F

* This problem was recently suggested to the writer by Professor L.L. Rauch.

assumes only the values*

$$\begin{aligned} K, \quad -(K + C) & \quad \left(\text{for } \frac{dx}{dt} > 0 \right) \quad \text{and} \\ -K, \quad K + C & \quad \left(\text{for } \frac{dx}{dt} < 0 \right) . \end{aligned}$$

This problem is mathematically equivalent with that which arises when it is assumed that the Coulomb damping is applied uninterruptedly, i.e., that $h \equiv +1$, and that $C < K$; in this case F takes on only the values*

$$\begin{aligned} K - C, \quad -K - C & \quad \left(\text{for } \frac{dx}{dt} > 0 \right) \quad \text{and} \\ K + C, \quad -K + C & \quad \left(\text{for } \frac{dx}{dt} < 0 \right) . \end{aligned}$$

Both problems are special cases of this one, which is a generalization of the problem on page 3:

For the equation

$$\frac{d^2x}{dt^2} + g\left(x, \frac{dx}{dt}\right) = \phi\left(x, \frac{dx}{dt}\right) \quad (41)$$

with g given and ϕ a function which is allowed to take on only the values $\alpha, -\beta$ (for $\frac{dx}{dt} > 0$), $-\alpha, \beta$ (for $\frac{dx}{dt} < 0$) (α and β being positive constants), how should ϕ be chosen so that every solution of (41) reaches the state $x = 0, \frac{dx}{dt} = 0$ in the least possible time?

The problem will be discussed here for the simplest case, $g \equiv 0$. This gives the system, equivalent with (41),

$$\frac{dx}{dt} = y, \quad \frac{dy}{dt} = \phi(x, y) . \quad (42)$$

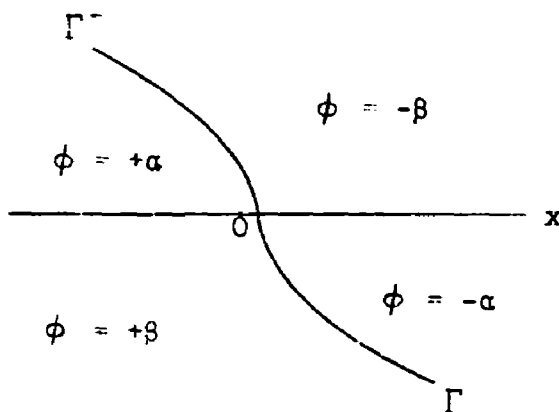
* What happens for $\frac{dx}{dt} = 0$ turns out not to matter; the solutions may be defined so as to extend across this axis as if F had the same values there as in a contiguous part of the plane.

Instead of talking about P- and N-curves, etc., we now speak of α , β , $-\alpha$, $-\beta$ -curves, etc. The considerations of sections I and III carry over in full, although the definition of solution (cf. pages 6-7) becomes even more awkward; in particular, Theorem 1, which states in effect that one needs to consider only canonical paths, still holds, where a canonical path is now defined as one which has no $(-\beta)(\alpha)$ -corners above $y = 0$ and no $(\beta)(-\alpha)$ -corners below. All this holds for any $g \in C^1$.

The α -curves for (42), in parameter-free form, are

$$y^2 = 2\alpha(x + k) \quad (-\infty < k < \infty)$$

and similarly for the other three values of ϕ . These are of course parabolas with $y = 0$ as their axis. If Γ is that part of the β -curve through the origin which lies below the x-axis, and Γ^- is that part of the $(-\beta)$ -curve through the origin which lies above, then by trivial extensions of the methods of IV (pages 26-28)



one gets that the unique minimal paths are obtained by taking ϕ as shown, with $\phi = \beta$ on Γ , $\phi = -\beta$ on Γ^- , $\phi = \alpha$ or β on the negative half of the x-axis, $\phi = -\alpha$ or $-\beta$ on the positive half. In fact, the analogous results for $g(x,y) = by$, $b \neq 0$, can also be proved in the same way. The problem for g less simple has not been studied.

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